OPTIMIZATION BY VECTOR SPACE METHODS
SERIES IN DECISION AND CONTROL

Ronald A. Howard

OPTIMIZATION BY VECTOR SPACE METHODS
  by David G. Luenberger

INTRODUCTION TO DYNAMIC PROGRAMMING
  by George L. Nemhauser

DYNAMIC PROBABILISTIC SYSTEMS (In Preparation)
  by Ronald A. Howard
OPTIMIZATION BY VECTOR SPACE METHODS

David G. Luenberger

Stanford University,
Stanford, California
To Nancy
This book has evolved from a course on optimization that I have taught at Stanford University for the past five years. It is intended to be essentially self-contained and should be suitable for classroom work or self-study. As a text it is aimed at first- or second-year graduate students in engineering, mathematics, operations research, or other disciplines dealing with optimization theory.

The primary objective of the book is to demonstrate that a rather large segment of the field of optimization can be effectively unified by a few geometric principles of linear vector space theory. By use of these principles, important and complex infinite-dimensional problems, such as those generated by consideration of time functions, are interpreted and solved by methods springing from our geometric insight. Concepts such as distance, orthogonality, and convexity play fundamental roles in this development. Viewed in these terms, seemingly diverse problems and techniques often are found to be intimately related.

The essential mathematical prerequisite is a familiarity with linear algebra, preferably from the geometric viewpoint. Some familiarity with elementary analysis including the basic notions of sets, convergence, and continuity is assumed, but deficiencies in this area can be corrected as one progresses through the book. More advanced concepts of analysis such as Lebesgue measure and integration theory, although referred to in a few isolated sections, are not required background for this book.

Imposing simple intuitive interpretations on complex infinite-dimensional problems requires a fair degree of mathematical sophistication. The backbone of the approach taken in this book is functional analysis, the study of linear vector spaces. In an attempt to keep the mathematical prerequisites to a minimum while not sacrificing completeness of the development, the early chapters of the book essentially constitute an introduction to functional analysis, with applications to optimization, for those having the relatively modest background described above. The mathematician or more advanced student may wish simply to scan Chapters 2, 3, 5, and 6 for review or for sections treating applications and then concentrate on the other chapters which deal explicitly with optimization theory.
The sequencing of the various sections is not necessarily inviolable. Even at the chapter level the reader may wish to alter his order of progress through the book. The course from which this text developed is two quarters (six months) in duration, but there is more material in the text than can be comfortably covered in that period. By reading only the first few sections of Chapter 3, it is possible to go directly from Chapters 1 and 2 to Chapters 5, 7, 8, and 10 for a fairly comprehensive treatment of optimization which can be covered in about one semester. Alternatively, the material at the end of Chapter 6 can be combined with Chapters 3 and 4 for a unified introduction to Hilbert space problems. To help the reader make intelligent decisions regarding his order of progress through the book, sections of a specialized or digressive nature are indicated by an *.

The problems at the end of each chapter are of two basic varieties. The first consists of miscellaneous mathematical problems and proofs which extend and supplement the theoretical material in the text; the second consists of optimization problems which illustrate further areas of application and which hopefully will help the student formulate and solve practical problems. The problems represent a major component of the book, and the serious student will not pass over them lightly.

I have received help and encouragement from many people during the years of preparation of this book. Of great benefit were comments and suggestions of Pravin Varaiya, E. Bruce Lee, and particularly Samuel Karlin who read the entire manuscript and suggested several valuable improvements. I wish to acknowledge the Departments of Engineering-Economic Systems and Electrical Engineering at Stanford University for supplying much of the financial assistance. This effort was also partially supported by the Office of Naval Research and the National Science Foundation. Of particular benefit, of course, have been the faces of puzzled confusion or of elated understanding, the critical comments and the sincere suggestions of the many students who have worked through this material as the book evolved.

DAVID G. LUENBERGER

Palo Alto, California
August 1968
CONTENTS

1 INTRODUCTION 1
   1.1 Motivation 1
   1.2 Applications 2
   1.3 The Main Principles 8
   1.4 Organization of the Book 10

2 LINEAR SPACES 11
   2.1 Introduction 11
       VECTOR SPACES 11
       2.2 Definition and Examples 11
       2.3 Subspaces, Linear Combinations, and Linear Varieties 14
       2.4 Convexity and Cones 17
       2.5 Linear Independence and Dimension 19
       NORMED LINEAR SPACES 22
       2.6 Definition and Examples 22
       2.7 Open and Closed Sets 24
       2.8 Convergence 26
       2.9 Transformations and Continuity 27
       *2.10 The $l_p$ and $L_p$ Spaces 29
       2.11 Banach Spaces 33
       2.12 Complete Subsets 38
       *2.13 Extreme Values of Functionals, and Compactness 39
       *2.14 Quotient Spaces 41
       *2.15 Denseness and Separability 42
       2.16 Problems 43
       References 45

3 HILBERT SPACE 46
   3.1 Introduction 46
## CONTENTS

### PRE-HILBERT SPACES 46

3.2 Inner Products 46  
3.3 The Projection Theorem 49  
3.4 Orthogonal Complements 52  
3.5 The Gram-Schmidt Procedure 53  

### APPROXIMATION 55

3.6 The Normal Equations and Gram Matrices 55  
3.7 Fourier Series 58  
*3.8 Complete Orthonormal Sequences 60  
3.9 Approximation and Fourier Series 62  

### OTHER MINIMUM NORM PROBLEMS 64

3.10 The Dual Approximation Problem 64  
*3.11 A Control Problem 68  
3.12 Minimum Distance to a Convex Set 69  
3.13 Problems 72  
References 77  

### LEAST-SQUARES ESTIMATION 78

4.1 Introduction 78  
4.2 Hilbert Space of Random Variables 79  
4.3 The Least-Squares Estimate 82  
4.4 Minimum-Variance Unbiased Estimate (Gauss-Markov Estimate) 84  
4.5 Minimum-Variance Estimate 87  
4.6 Additional Properties of Minimum-Variance Estimates 90  
4.7 Recursive Estimation 93  
4.8 Problems 97  
References 102  

### DUAL SPACES 103

5.1 Introduction 103  

#### LINEAR FUNCTIONALS 104

5.2 Basic Concepts 104  
5.3 Duals of Some Common Banach Spaces 106  

#### EXTENSION FORM OF THE HAHN-BANACH THEOREM 110

5.4 Extension of Linear Functionals 110  
5.5 The Dual of \( C[a, b] \) 113  
5.6 The Second Dual Space 115  
5.7 Alignment and Orthogonal Complements 116
CONTENTS xi

5.8 Minimum Norm Problems 118
5.9 Applications 122
*5.10 Weak Convergence 126

GEOMETRIC FORM OF THE HAHN-BANACH THEOREM 129

5.11 Hyperplanes and Linear Functionals 129
5.12 Hyperplanes and Convex Sets 131
*5.13 Duality in Minimum Norm Problems 134
5.14 Problems 137
References 142

6 LINEAR OPERATORS AND ADJOINTS 143

6.1 Introduction 143
6.2 Fundamentals 143

INVERSE OPERATORS 147
6.3 Linearity of Inverses 147
6.4 The Banach Inverse Theorem 148

ADJOINTS 150
6.5 Definition and Examples 150
6.6 Relations between Range and Nullspace 155
6.7 Duality Relations for Convex Cones 157
*6.8 Geometric Interpretation of Adjoint 159

OPTIMIZATION IN HILBERT SPACE 160
6.9 The Normal Equations 160
6.10 The Dual Problem 161
6.11 Pseudoinverse Operators 163
6.12 Problems 165
References 168

7 OPTIMIZATION OF FUNCTIONALS 169

7.1 Introduction 169

LOCAL THEORY 171
7.2 Gateaux and Fréchet Differentials 171
7.3 Fréchet Derivatives 175
7.4 Extrema 177
*7.5 Euler-Lagrange Equations 179
*7.6 Problems with Variable End Points 183
7.7 Problems with Constraints 185
GLOBAL THEORY 190
7.8 Convex and Concave Functionals 190
*7.9 Properties of the Set \([f, C]\) 192
7.10 Conjugate Convex Functionals 195
7.11 Conjugate Concave Functionals 199
7.12 Dual Optimization Problems 200
*7.13 Min-Max Theorem of Game Theory 206
7.14 Problems 209
References 212

8 GLOBAL THEORY OF CONSTRAINED OPTIMIZATION 213
8.1 Introduction 213
8.2 Positive Cones and Convex Mappings 214
8.3 Lagrange Multipliers 216
8.4 Sufficiency 219
8.5 Sensitivity 221
8.6 Duality 223
8.7 Applications 226
8.8 Problems 236
References 238

9 LOCAL THEORY OF CONSTRAINED OPTIMIZATION 239
9.1 Introduction 239

LAGRANGE MULTIPLIER THEOREMS 240
9.2 Inverse Function Theorem 240
9.3 Equality Constraints 242
9.4 Inequality Constraints (Kuhn-Tucker Theorem) 247

OPTIMAL CONTROL THEORY 254
9.5 Basic Necessary Conditions 254
*9.6 Pontryagin Maximum Principle 261
9.7 Problems 266
References 269

10 ITERATIVE METHODS OF OPTIMIZATION 271
10.1 Introduction 271

METHODS FOR SOLVING EQUATIONS 272
10.2 Successive Approximation 272
10.3 Newton's Method 277
NOTATION

Sets

If \( x \) is a member of the set \( S \), we write \( x \in S \). The notation \( y \notin S \) means \( y \) is not a member of \( S \).

A set may be specified by listing its elements between braces such as \( S = \{1, 2, 3\} \) for the set consisting of the first three positive integers. Alternatively, a set \( S \) may be specified as consisting of all elements of the set \( X \) which have the property \( P \). This is written \( S = \{x \in X: P(x)\} \) or, if \( X \) is understood, \( S = \{x: P(x)\} \).

The union of two sets \( S \) and \( T \) is denoted \( S \cup T \) and consists of those elements that are in either \( S \) or \( T \).

The intersection of two sets \( S \) and \( T \) is denoted \( S \cap T \) and consists of those elements that are in both \( S \) and \( T \). Two sets are disjoint if their intersection is empty.

If \( S \) is defined as a subset of elements of \( X \), the complement of \( S \), denoted \( \bar{S} \), consists of those elements of \( X \) that are not in \( S \).

A set \( S \) is a subset of the set \( T \) if every element of \( S \) is also an element of \( T \). In this case we write \( S \subseteq T \) or \( T \supseteq S \). If \( S \subseteq T \) and \( S \) is not equal to \( T \) then \( S \) is said to be a proper subset of \( T \).

Sets of Real Numbers

If \( a \) and \( b \) are real numbers, \( [a, b] \) denotes the set of real numbers \( x \) satisfying \( a \leq x \leq b \). A rounded instead of square bracket denotes strict inequality in the definition. Thus \( (a, b] \) denotes all \( x \) with \( a < x \leq b \).

If \( S \) is a set of real numbers bounded above, then there is a smallest real number \( y \) such that \( x \leq y \) for all \( x \in S \). The number \( y \) is called the least upper bound or supremum of \( S \) and is denoted \( \sup_{x \in S} \) or \( \{ x: x \in S \} \). If \( S \) is not bounded above we write \( \sup_{x \in S} (x) = \infty \). Similarly, the greatest lower bound or infimum of a set \( S \) is denoted \( \inf_{x \in S} \) or \( \{ x: x \in S \} \).
Sequences

A sequence \( x_1, x_2, \ldots, x_n, \ldots \) is denoted by \( \{x_i\}_{i=1}^{\infty} \) or \( \{x_i\} \) if the range of the indices is clear.

Let \( \{x_i\} \) be an infinite sequence of real numbers and suppose that there is a real number \( S \) satisfying: (1) for every \( \varepsilon > 0 \) there is an \( N \) such that for all \( n > N, x_n < S + \varepsilon \), and (2) for every \( \varepsilon > 0 \) and \( M > 0 \) there is an \( n > M \) such that \( x_n > S - \varepsilon \). Then \( S \) is called the limit superior of \( \{x_n\} \) and we write \( S = \limsup_{n \to \infty} x_n \). If \( \{x_n\} \) is not bounded above we write \( \limsup_{n \to \infty} x_n = +\infty \).

The limit inferior of \( x_n \) is \( \liminf_{n \to \infty} x_n = -\limsup_{n \to \infty} (-x_n) \). If \( \limsup_{n \to \infty} x_n = \liminf_{n \to \infty} x_n = S \), we write \( \lim_{n \to \infty} x_n = S \).

Functions

The function \( \text{sgn} \) (pronounced sig-num) of a real variable is defined by

\[
\text{sgn}(x) = \begin{cases} 
1 & x > 0 \\
0 & x = 0 \\
-1 & x < 0 
\end{cases}
\]

The Kronecker delta function \( \delta_{ij} \) is defined by

\[
\delta_{ij} = \begin{cases} 
1 & i = j \\
0 & i \neq j 
\end{cases}
\]

The Dirac delta function \( \delta \) is used occasionally in heuristic discussions. It is defined by the relation

\[
\int_a^b f(t)\delta(t) \, dt = f(0)
\]

for every continuous function \( f \) provided that \( 0 \in (a, b) \).

If \( g \) is a real-valued function of a real variable we write \( S = \limsup_{x \to x_0} g(x) \) if: (1) for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that for all \( x \) satisfying \( |x - x_0| < \delta \), \( g(x) < S + \varepsilon \), and (2) for every \( \varepsilon > 0 \) and \( \delta > 0 \) there is an \( x \) such that \( |x - x_0| < \delta \) and \( g(x) > S - \varepsilon \). (See the corresponding definitions for sequences.)

If \( g \) is a real-valued function of a real variable, the notation \( g(x) = O(x) \) means that

\[
K = \limsup_{x \to 0} \left| \frac{g(x)}{x} \right|
\]

is finite. The notation \( g(x) = o(x) \) means that \( K \), above, is zero.
Matrices and Vectors

A vector $x$ with $n$ components is written $x = (x_1, x_2, \ldots, x_n)$, but when used in matrix calculations it is represented as a column vector, i.e.,

$$
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}
$$

The corresponding row vector is

$$
x' = x_1 \ x_2 \cdots x_n
$$

An $n \times m$ matrix $A$ with entry $a_{ij}$ in its $i$-th row and $j$-th column is written $A = [a_{ij}]$. If $x = (x_1, x_2, \ldots, x_n)$, the product $Ax$ is the vector $y$ with components $y_i = \sum_{j=1}^{n} a_{ij} x_j$, $i = 1, 2, \ldots, m$.

Let $f(x_1, x_2, \ldots, x_n)$ be a function of the $n$ real variables $x_i$. Then we write $f_x$ for the row vector

$$
\begin{bmatrix}
  \frac{\partial f}{\partial x_1} \\
  \frac{\partial f}{\partial x_2} \\
  \vdots \\
  \frac{\partial f}{\partial x_n}
\end{bmatrix}
$$

If $F = (f_1, f_2, \ldots, f_m)$ is a vector function of $x = (x_1, \ldots, x_n)$, we write $F_x$ for the $m \times n$ Jacobian matrix $[\partial f_i/\partial x_j]$.
1.1 Motivation

During the past twenty years mathematics and engineering have been increasingly directed towards problems of decision making in physical or organizational systems. This trend has been inspired primarily by the significant economic benefits which often result from a proper decision concerning the distribution of expensive resources, and by the repeated demonstration that such problems can be realistically formulated and mathematically analyzed to obtain good decisions.

The arrival of high-speed digital computers has also played a major role in the development of the science of decision making. Computers have inspired the development of larger systems and the coupling of previously separate systems, thereby resulting in decision and control problems of correspondingly increased complexity. At the same time, however, computers have revolutionized applied mathematics and solved many of the complex problems they generated.

It is perhaps natural that the concept of best or optimal decisions should emerge as the fundamental approach for formulating decision problems. In this approach a single real quantity, summarizing the performance or value of a decision, is isolated and optimized (i.e., either maximized or minimized depending on the situation) by proper selection among available alternatives. The resulting optimal decision is taken as the solution to the decision problem. This approach to decision problems has the virtues of simplicity, preciseness, elegance, and, in many cases, mathematical tractability. It also has obvious limitations due to the necessity of selecting a single objective by which to measure results. But optimization has proved its utility as a mode of analysis and is firmly entrenched in the field of decision making.

Much of the classical theory of optimization, motivated primarily by problems of physics, is associated with great mathematicians: Gauss, Lagrange, Euler, the Bernoullis, etc. During the recent development of optimization in decision problems, the classical techniques have been re-examined, extended, sometimes rediscovered, and applied to problems.
2 INTRODUCTION

having quite different origins than those responsible for their earlier development. New insights have been obtained and new techniques have been discovered. The computer has rendered many techniques obsolete while making other previously impractical methods feasible and efficient. These recent developments in optimization have been made by mathematicians, system engineers, economists, operations researchers, statisticians, numerical analysts, and others in a host of different fields.

The study of optimization as an independent topic must, of course, be regarded as a branch of applied mathematics. As such it must look to various areas of pure mathematics for its unification, clarification, and general foundation. One such area of particular relevance is functional analysis.

Functional analysis is the study of vector spaces resulting from a merging of geometry, linear algebra, and analysis. It serves as a basis for aspects of several important branches of applied mathematics including Fourier series, integral and differential equations, numerical analysis, and any field where linearity plays a key role. Its appeal as a unifying discipline stems primarily from its geometric character. Most of the principal results in functional analysis are expressed as abstractions of intuitive geometric properties of ordinary three-dimensional space.

Some readers may look with great expectation toward functional analysis, hoping to discover new powerful techniques that will enable them to solve important problems beyond the reach of simpler mathematical analysis. Such hopes are rarely realized in practice. The primary utility of functional analysis for the purposes of this book is its role as a unifying discipline, gathering a number of apparently diverse, specialized mathematical tricks into one or a few general geometric principles.

1.2 Applications

The main purpose of this section is to illustrate the variety of problems that can be formulated as optimization problems in vector space by introducing some specific examples that are treated in later chapters. As a vehicle for this purpose, we classify optimization problems according to the role of the decision maker. We list the classification, briefly describe its meaning, and illustrate it with one problem that can be formulated in vector space and treated by the methods described later in the book. The classification is not intended to be necessarily complete nor, for that matter, particularly significant. It is merely representative of the classifications often employed when discussing optimization.

Although the formal definition of a vector space is not given until Chapter 2, we point out, in the examples that follow, how each problem
can be regarded as formulated in some appropriate vector space. However, the details of the formulation must, in many cases, be deferred until later chapters.

1. Allocation. In allocation problems there is typically a collection of resources to be distributed in some optimal fashion. Almost any optimization problem can be placed in this broad category, but usually the term is reserved for problems in which the resources are distributed over space or among various activities.

A typical problem of this type is that faced by a manufacturer with an inventory of raw materials. He has certain processing equipment capable of producing \( n \) different kinds of goods from the raw materials. His problem is to allocate the raw materials among the possible products so as to maximize his profit.

In an idealized version of the problem, we assume that the production and profit model is linear. Assume that the selling price per unit of product \( j \) is \( p_j, j = 1, 2, \ldots, n \). If \( x_j \) denotes the amount of product \( j \) that is to be produced, \( b_i \) the amount of raw material \( i \) on hand, and \( a_{ij} \) the amount of material \( i \) in one unit of product \( j \), the manufacturer seeks to maximize his profit

\[
p_1 x_1 + p_2 x_2 + \cdots + p_n x_n
\]

subject to the production constraints on the amount of raw materials

\[
a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \leq b_i
\]

\[
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2
\]

\[
\vdots
\]

\[
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m
\]

and

\[
x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0.
\]

This type of problem, characterized by a linear objective function subject to linear inequality constraints, is a linear programming problem and is used to illustrate aspects of the general theory of optimization in later chapters.

We note that the problem can be regarded as formulated in ordinary \( n \)-dimensional vector space. The vector \( x \) with components \( x_i \) is the unknown. The constraints define a region in the vector space in which the selected vector must lie. The optimal vector is the one in that region maximizing the objective.

The manufacturing problem can be generalized to allow for nonlinear
objectives and more general constraints. Linearity is destroyed, which may make the solution more difficult to obtain, but the problem can still be regarded as one in ordinary Euclidean n-dimensional space.

2. Planning. Planning is the problem of determining an optimal procedure for attaining a set of objectives. In common usage, planning refers especially to those problems involving outlays of capital over a period of time such as (1) planning a future investment in electric power generation equipment for a given geographic region or (2) determining the best hiring policy in order to complete a complex project at minimum expense.

As an example, consider a problem of production planning. A firm producing a certain product wishes to plan its production schedule over a period of time in an optimal fashion. It is assumed that a fixed demand function over the time interval is known and that this demand must be met. Excess inventory must be stored at a storage cost proportional to the amount stored. There is a production cost associated with a given rate of production. Thus, denoting \( x(t) \) as the stock held at time \( t \), \( r(t) \) as the rate of production at time \( t \), and \( d(t) \) as the demand at time \( t \), the production system can be described by the equations

\[
\begin{align*}
x(t) &= r(t) - d(t), \quad x(0) \text{ given} \\
\int_0^T [r(t) - d(t)] \, dt &= x(t) \\
\int_0^T c[r(t)] + h \cdot x(t) \, dt &= J
\end{align*}
\]

and one seeks the function \( r \) satisfying the inequality constraints

\[
\begin{align*}
\int_0^T [r(t) - d(t)] \, dt &= x(t) \geq 0 \\
\int_0^T c[r(t)] + h \cdot x(t) \, dt &= J
\end{align*}
\]

and minimizing the cost

\[
J = \int_0^T c[r(t)] + h \cdot x(t) \, dt
\]

where \( c[r] \) is the production cost rate for the production level \( r \) and \( h \cdot x \) is the inventory cost rate for inventory level \( x \).

This problem can be regarded as defined on a vector space consisting of continuous functions on the interval \([0, T]\) of the real line. The optimal production schedule \( r \) is then an element of the space. Again the constraints define a region in the space in which the solution \( r \) must lie while minimizing the cost.

3. Control (or Guidance). Problems of control are associated with dynamic systems evolving in time. Control is quite similar to planning;

\[ \dot{x}(t) = dx(t)/dt. \]

\[ \]
indeed, as we shall see, it is often the source of a problem rather than its mathematical structure which determines its category.

Control or guidance usually refers to directed influence on a dynamic system to achieve desired performance. The system itself may be physical in nature, such as a rocket heading for Mars or a chemical plant processing acid, or it may be operational such as a warehouse receiving and filling orders.

Often we seek feedback or so-called closed-loop control in which decisions of current control action are made continuously in time based on recent observations of system behavior. Thus, one may imagine himself as a controller sitting at the control panel watching meters and turning knobs or in a warehouse ordering new stock based on inventory and predicted demand. This is in contrast to the approach described for planning in which the whole series of control actions is predetermined. Generally, however, the terms planning or control may refer to either possibility.

As an example of a control problem, we consider the launch of a rocket to a fixed altitude \( h \) in given time \( T \) while expending a minimum of fuel. For simplicity, we assume unit mass, a constant gravitational force, and the absence of aerodynamic forces. The motion of a rocket being propelled vertically is governed by the equations

\[
y(t) = u(t) - g
\]

where \( y \) is the vertical height, \( u \) is the accelerating force, and \( g \) is the gravitational force. The optimal control function \( u \) is the one which forces \( y(T) = h \) while minimizing the fuel expenditure \( \int_0^T |u(t)| \, dt \).

This problem too might be formulated in a vector space consisting of functions \( u \) defined on the interval \([0, T]\). The solution to this problem, however, is that \( u(t) \) consists of an impulse at \( t = 0 \) and, therefore, correct problem formulation and selection of an appropriate vector space are themselves interesting aspects of this example. Problems of this type, including this specific example, are discussed in Chapter 5.

4. Approximation. Approximation problems are motivated by the desire to approximate a general mathematical entity (such as a function) by one of simpler, specified form. A large class of such approximation problems is important in numerical analysis. For example, suppose we wish, because of storage limitations or for purposes of simplifying an analysis, to approximate a function, say \( x(t) \), over an interval \([a, b]\) of the real line by a polynomial \( p(t) \) of order \( n \). The best approximating polynomial \( p \) minimizes the error \( e = x - p \) in the sense of some criterion. The choice of criterion determines the approximation. Often used criteria are:
1. \[ \int_{a}^{b} e^{2}(t) \, dt \]

2. \[ \max_{a \leq t \leq b} |e(t)| \]

3. \[ \int_{a}^{b} |e(t)| \, dt. \]

The problem is quite naturally viewed as formulated in a vector space of functions over the interval \([a, b]\). The problem is then viewed as finding a vector from a given class (polynomials) which is closest to a given vector.

5. **Estimation.** Estimation problems are really a special class of approximation problems. We seek to estimate some quantity from imperfect observations of it or from observations which are statistically correlated but not deterministically related to it. Loosely speaking, the problem amounts to approximating the unobservable quantity by a combination of the observable ones. For example, the position of a random maneuvering airplane at some future time might reasonably be estimated by a linear combination of past measurements of its position.

Another example of estimation arises in connection with triangulation problems such as in location of forest fires, ships at sea, or remote stars. Suppose there are three lookout stations, each of which measures the angle of the line-of-sight from the station to the observed object. The situation is illustrated in Figure 1.1. Given these three angles, what is the best estimate of the object's location?

![Figure 1.1 A triangulation problem](image_url)

To formulate the problem completely, a criterion must be precisely prescribed and hypotheses specified regarding the nature of probable
measurement errors and probable location of the object. Approaches can be taken that result in a problem formulated in vector space; such problems are discussed in Chapter 4.

6. Games. Many problems involving a competitive element can be regarded as games. In the usual formulation, involving two players or protagonists, there is an objective function whose value depends jointly on the action employed by both players. One player attempts to maximize this objective while the other attempts to minimize it.

Often two problems from the categories discussed above can be competitively intermixed to produce a game. Combinations of categories that lead to interesting games include: allocation-allocation, allocation-control, control-control, and estimation-control.

As an example, consider a control-control game. Most problems of this type are of the pursuer-evader type such as a fighter plane chasing a bomber. Each player has a system he controls but one is trying to maximize the objective (time to intercept for instance) while the other is trying to minimize the objective.

As a simpler example, we consider a problem of advertising or campaigning which is essentially an allocation-allocation game. Two opposing candidates, $A$ and $B$, are running for office and must plan how to allocate their advertising resources ($A$ and $B$ dollars, respectively) among $n$ distinct geographical areas. Let $x_i$ and $y_i$ denote, respectively, the resources committed to area $i$ by candidates $A$ and $B$. We assume that there are currently a total of $u$ undecided votes of which there are $u_i$ undecided votes in area $i$. The number of votes going to candidates $A$ and $B$ from area $i$ are assumed to be

\[
\frac{x_i u_i}{x_i + y_i}, \quad \frac{y_i u_i}{x_i + y_i}
\]

respectively. The total difference between the number of votes received by $A$ and by $B$ is then

\[
\sum_{i=1}^{n} \frac{x_i - y_i}{x_i + y_i} u_i.
\]

Candidate $A$ seeks to maximize this quantity while $B$ seeks to minimize it.

This problem is obviously finite dimensional and can be solved by ordinary calculus in a few lines. It is illustrative, however, of an interesting class of game problems.

2 This problem is due to L. Friedman [57].
1.3 The Main Principles

The theory of optimization presented in this book is derived from a few simple, intuitive, geometric relations. The extension of these relations to infinite-dimensional spaces is the motivation for the mathematics of functional analysis which, in a sense, often enables us to extend our three-dimensional geometric insights to complex infinite-dimensional problems. This is the conceptual utility of functional analysis. On the other hand, these simple geometric relations have great practical utility as well because a vast assortment of problems can be analyzed from this point of view.

In this section, we briefly describe a few of the important geometric principles of optimization that are developed in detail in later chapters.

1. The Projection Theorem. This theorem is one of the simplest and nicest results of optimization theory. In ordinary three-dimensional Euclidean space, it states that the shortest line from a point to a plane is furnished by the perpendicular from the point to the plane, as illustrated in Figure 1.2.

![Figure 1.2 The projection theorem](image)

This simple and seemingly innocuous result has direct extensions in spaces of higher dimension and in infinite-dimensional Hilbert space. In the generalized form, this optimization principle forms the basis of all least-squares approximation, control, and estimation procedures.

2. The Hahn-Banach Theorem. Of the many results and concepts in functional analysis, the one theorem dominating the theme of this book and embodying the essence of the simple geometric ideas upon which the theory is built is the Hahn-Banach theorem. The theorem takes several forms. One version extends the projection theorem to problems having nonquadratic objectives. In this manner the simple geometric interpretation is preserved for these more complex problems. Another version of the
Hahn-Banach theorem states (in simplest form) that given a sphere and a point not in the sphere there is a hyperplane separating the point and the sphere. This version of the theorem, together with the associated notions of hyperplanes, is the basis for most of the theory beyond Chapter 5.

3. **Duality.** There are several duality principles in optimization theory that relate a problem expressed in terms of vectors in a space to a problem expressed in terms of hyperplanes in the space. This concept of duality is a recurring theme in this book.

Many of these duality principles are based on the geometric relation illustrated in Figure 1.3. The shortest distance from a point to a convex set is equal to the maximum of the distances from the point to a hyperplane separating the point from the convex set. Thus, the original minimization over vectors can be converted to maximization over hyperplanes.

![Figure 1.3 Duality](image)

4. **Differentials.** Perhaps the most familiar optimization technique is the method of differential calculus—setting the derivative of the objective function equal to zero. The technique is discussed for a single or, perhaps, finite number of variables in the most elementary courses on differential calculus. Its extension to infinite-dimensional spaces is straightforward and, in that form, it can be applied to a variety of interesting optimization problems. Much of the classical theory of the calculus of variations can be viewed as a consequence of this principle.

The geometric interpretation of the technique for one-dimensional problems is obvious. At a maximum or minimum the tangent to the graph of a function is horizontal. In higher dimensions the geometric interpretation is similar: at a maximum or minimum the tangent hyperplane to the
graph is horizontal. Thus, again we are led to observe the fundamental role of hyperplanes in optimization.

1.4 Organization of the Book

Before our discussion of optimization can begin in earnest, certain fundamental concepts and results of linear vector space theory must be introduced. Chapter 2 is devoted to that task. The chapter consists of material that is standard, elementary functional analysis background and is essential for further pursuit of our objectives. Anyone having some familiarity with linear algebra and analysis should have little difficulty with this chapter.

Chapters 3 and 4 are devoted to the projection theorem in Hilbert space and its applications. Chapter 3 develops the general theory, illustrating it with some applications from Fourier approximation and optimal control theory. Chapter 4 deals solely with the applications of the projection theorem to estimation problems including the recursive estimation and prediction of time series as developed by Kalman.

Chapter 5 is devoted to the Hahn-Banach theorem. It is in this chapter that we meet with full force the essential ingredients of the general theory of optimization: hyperplanes, duality, and convexity.

Chapter 6 discusses linear transformations on a vector space and is the last chapter devoted to the elements of linear functional analysis. The concept of duality is pursued in this chapter through the introduction of adjoint transformations and their relation to minimum norm problems. The pseudoinverse of an operator in Hilbert space is discussed.

Chapters 7, 8, and 9 consider general optimization problems in linear spaces. Two basic approaches, the local theory leading to differential conditions and the global theory relying on convexity, are isolated and discussed in a parallel fashion. The techniques in these chapters are a direct outgrowth of the principles of earlier chapters, and geometric visualization is stressed wherever possible. In the course of the development, we treat problems from the calculus of variations, the Fenchel conjugate function theory, Lagrange multipliers, the Kuhn-Tucker theorem, and Pontryagin's maximum principle for optimal control problems.

Finally, Chapter 10 contains an introduction to iterative techniques for the solution of optimization problems. Some techniques in this chapter are quite different than those in previous chapters, but many are based on extensions of the same logic and geometrical considerations found to be so fruitful throughout the book. The methods discussed include successive approximation, Newton’s method, steepest descent, conjugate gradients, the primal-dual method, and penalty functions.
2

LINEAR SPACES

2.1 Introduction

The first few sections of this chapter introduce the concept of a vector space and explore the elementary properties resulting from the basic definition. The notions of subspace, linear independence, convexity, and dimension are developed and illustrated by examples. The material is largely review for most readers since it duplicates the first part of standard courses in linear algebra.

The second part of the chapter discusses the basic properties of normed linear spaces. A normed linear space is a vector space having a measure of distance or length defined on it. With the introduction of a norm, it becomes possible to define analytical or topological properties such as convergence and open and closed sets. Therefore, that portion of the chapter introduces and explores these basic concepts which distinguish functional analysis from linear algebra.

VECTOR SPACES

2.2 Definition and Examples

Associated with every vector space is a set of scalars used to define scalar multiplication on the space. In the most abstract setting these scalars are required only to be elements of an algebraic field. However, in this book the scalars are always taken to be either the set of real numbers or of complex numbers. We sometimes distinguish between these possibilities by referring to a vector space as either a real or a complex vector space. In this book, however, the primary emphasis is on real vector spaces and, although occasional reference is made to complex spaces, many results are derived only for real spaces. In case of ambiguity, the reader should assume the space to be real.

Definition. A vector space \( X \) is a set of elements called vectors together with two operations. The first operation is addition which associates with
any two vectors \( x, y \in X \) a vector \( x + y \in X \), the sum of \( x \) and \( y \). The second operation is scalar multiplication which associates with any vector \( x \in X \) and any scalar \( \alpha \) a vector \( \alpha x \); the scalar multiple of \( x \) by \( \alpha \). The set \( X \) and the operations of addition and scalar multiplication are assumed to satisfy the following axioms:

1. \( x + y = y + x \).  
   (commutative law)
2. \( (x + y) + z = x + (y + z) \).  
   (associative law)
3. There is a null vector \( \theta \) in \( X \) such that \( x + \theta = x \) for all \( x \) in \( X \).
4. \( \alpha(x + y) = \alpha x + \alpha y \).  
   (distributive laws)
5. \( (\alpha + \beta)x = \alpha x + \beta x \).  
   (associative law)
6. \( (\alpha\beta)x = \alpha(\beta x) \).
7. \( 0x = \theta, \quad 1x = x \).

For convenience the vector \(-1x\) is denoted \(-x\) and called the negative of the vector \( x \). We have \( x + (-x) = (1 - 1)x = 0x = \theta \).

There are several elementary but important properties of vector spaces that follow directly from the axioms listed in the definition. For example, the following properties are easily deduced. The details are left to the reader.

**Proposition 1.** In any vector space:

1. \( x + y = x + z \) implies \( y = z \).  
   (cancellation laws)
2. \( \alpha x = \alpha y \) and \( \alpha \neq 0 \) imply \( x = y \).
3. \( \alpha x = \beta x \) and \( x \neq \theta \) imply \( \alpha = \beta \).
4. \( (\alpha - \beta)x = \alpha x - \beta x \).  
   (distributive laws)
5. \( \alpha(x - y) = \alpha x - \alpha y \).
6. \( \alpha \theta = \theta \).

Some additional properties are given as exercises at the end of the chapter.

**Example 1.** Perhaps the simplest example of a vector space is the set of real numbers. It is a real vector space with addition defined in the usual way and multiplication by (real) scalars defined as ordinary multiplication. The null vector is the real number zero. The properties of ordinary addition and multiplication of real numbers satisfy the axioms in the definition of a vector space. This vector space is called the one-dimensional real coordinate space or simply the real line. It is denoted by \( R^1 \) or simply \( R \).

**Example 2.** An obvious extension of Example 1 is to \( n \)-dimensional real coordinate space. Vectors in the space consist of sequences (\( n \)-tuples) of \( n \) real numbers so that a typical vector has the form \( x = (\xi_1, \xi_2, \ldots, \xi_n) \).
The real number $\xi_k$ is referred to as the $k$-th component of the vector. Two vectors are equal if their corresponding components are equal. The null vector is defined as $\theta = (0, 0, \ldots, 0)$. If $x = (\xi_1, \xi_2, \ldots, \xi_n)$ and $y = (\eta_1, \eta_2, \ldots, \eta_n)$, the vector $x + y$ is defined as the $n$-tuple whose $k$-th component is $\xi_k + \eta_k$. The vector $\alpha x$, where $\alpha$ is a (real) scalar, is the $n$-tuple whose $k$-th component is $\alpha \xi_k$. The axioms in the definition are verified by checking for equality among components. For example, if $x = (\xi_1, \xi_2, \ldots, \xi_n)$, the relation $\xi_k + 0 = \xi_k$ implies $x + \theta = x$.

This space, $n$-dimensional real coordinate space, is denoted by $R^n$. The corresponding complex space consisting of $n$-tuples of complex numbers is denoted by $C^n$.

At this point we are, strictly speaking, somewhat prematurely introducing the term dimensionality. Later in this chapter the notion of dimension is defined, and it is proved that these spaces are in fact $n$ dimensional.

**Example 3.** Several interesting vector spaces can be constructed with vectors consisting of infinite sequences of real numbers so that a typical vector has the form $x = (\xi_1, \xi_2, \ldots, \xi_k, \ldots)$ or, equivalently, $x = \{\xi_k\}_{k=1}^\infty$. Again addition and multiplication are defined componentwise as in Example 2. The collection of all infinite sequences of real numbers forms a vector space. A sequence $\{\xi_k\}$ is said to be bounded if there is a constant $M$ such that $|\xi_k| < M$ for all $k$. The collection of all bounded infinite sequences forms a vector space since the sum of two bounded sequences or the scalar multiple of a bounded sequence is again bounded. This space is referred to as the space of bounded real sequences.

**Example 4.** The collection of all sequences of real numbers having only a finite number of terms not equal to zero is a vector space. (Different members of the space may have different numbers of nonzero components.) This space is called the space of finitely nonzero sequences.

**Example 5.** The collection of infinite sequences of real numbers which converge to zero is a vector space since the sum of two sequences converging to zero or the scalar multiple of a sequence converging to zero also converges to zero.

**Example 6.** Consider the interval $[a, b]$ on the real line. The collection of all real-valued continuous functions on this interval forms a vector space. Write $x = y$ if $x(t) = y(t)$ for all $t \in [a, b]$. The null vector $\theta$ is the function identically zero on $[a, b]$. If $x$ and $y$ are vectors in the space and $\alpha$ is a (real) scalar, write $(x + y)(t) = x(t) + y(t)$ and $(\alpha x)(t) = \alpha x(t)$. These are obviously continuous functions. This space is referred to as the vector space of real-valued continuous functions on $[a, b]$. 


Example 7. The collection of all polynomial functions defined on the interval \([a, b]\) with complex coefficients forms a complex vector space. The null vector, addition, and scalar multiplication are defined as in Example 6. The sum of two polynomials and the scalar multiple of a polynomial are themselves polynomials.

We now consider how a set of vector spaces can be combined to produce a larger one.

Definition. Let \(X\) and \(Y\) be vector spaces over the same field of scalars. Then the Cartesian product of \(X\) and \(Y\), denoted \(X \times Y\), consists of the collection of ordered pairs \((x, y)\) with \(x \in X, y \in Y\). Addition and scalar multiplication are defined on \(X \times Y\) by \((x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)\) and \(a(x, y) = (ax, ay)\).

That the above definition is consistent with the axioms of a vector space is obvious. The definition is easily generalized to the product of \(n\) vector spaces \(X_1, X_2, \ldots, X_n\). We write \(X^n\) for the product of a vector space with itself \(n\) times.

2.3 Subspaces, Linear Combinations, and Linear Varieties

Definition. A nonempty subset \(M\) of a vector space \(X\) is called a subspace of \(X\) if every vector of the form \(\alpha x + \beta y\) is in \(M\) whenever \(x\) and \(y\) are both in \(M\).

Since a subspace is assumed to be nonempty, it must contain at least one element \(x\). By definition, it must then also contain \(0x = \theta\), so every subspace contains the null vector. The simplest subspace is the set consisting of \(\theta\) alone. In three-dimensional space a plane that passes through the origin is a subspace as is a line through the origin.

The entire space \(X\) is itself a subspace of \(X\). A subspace not equal to the entire space is said to be a proper subspace.

Any subspace contains sums and scalar multiples of its elements; satisfaction of the seven axioms in the entire space implies that they are satisfied within the subspace. Therefore a subspace is itself a vector space and this observation justifies the terminology.

If \(X\) is the space of \(n\)-tuples, the set of \(n\)-tuples having the first component equal to zero is a subspace of \(X\). The space of convergent infinite sequences is a subspace of the vector space of bounded sequences. The space of continuous functions on \([0, 1]\) which are zero at the point one-half is a subspace of the vector space of continuous functions.

In three-dimensional space, the intersection of two distinct planes
containing the origin is a line containing the origin. This is a special case of the following result.

**Proposition 1.** Let $M$ and $N$ be subspaces of a vector space $X$. Then the intersection, $M \cap N$, of $M$ and $N$ is a subspace of $X$.

*Proof.* $M \cap N$ contains $\theta$ since $\theta$ is contained in each of the subspaces $M$ and $N$. Therefore, $M \cap N$ is nonempty. If $x$ and $y$ are in $M \cap N$, they are in both $M$ and $N$. For any scalars $\alpha$, $\beta$ the vector $\alpha x + \beta y$ is contained in both $M$ and $N$ since $M$ and $N$ are subspaces. Therefore, $\alpha x + \beta y$ is contained in the intersection $M \cap N$.

The union of two subspaces in a vector space is not necessarily a subspace. In the plane, for example, the union of two (noncolinear) lines through the origin does not contain arbitrary sums of its elements. However, two subspaces may be combined to form a larger subspace by introducing the notion of the sum of two sets.

**Definition.** The *sum* of two subsets $S$ and $T$ in a vector space, denoted $S + T$, consists of all vectors of the form $s + t$ where $s \in S$ and $t \in T$.

The sum of two sets is illustrated in Figure 2.1.

![Figure 2.1 The sum of two sets](image)

**Proposition 2.** Let $M$ and $N$ be subspaces of a vector space $X$. Then their sum, $M + N$, is a subspace of $X$.

*Proof.* Clearly $M + N$ contains $\theta$. Suppose $x$ and $y$ are vectors in $M + N$. There are vectors $m_1$, $m_2$ in $M$ and vectors $n_1$, $n_2$ in $N$ such that
\[ x = m_1 + n_1, \ y = m_2 + n_2. \] For any scalars \( \alpha, \beta; \alpha x + \beta y = (\alpha m_1 + \beta m_2) + (\alpha n_1 + \beta n_2). \) Therefore, \( \alpha x + \beta y \) can be expressed as the sum of a vector in the subspace \( M \) and a vector in the subspace \( N. \]

In two-dimensional Euclidean space the sum of two noncolinear lines through the origin is the entire space. The set of even continuous functions and the set of odd continuous functions are subspaces of the space \( X \) of continuous functions on the real line. Their sum is the whole space \( X \) since any continuous function can be expressed as a sum of an even and an odd continuous function.

**Definition.** A linear combination of the vectors \( x_1, x_2, \ldots, x_n \) in a vector space is a sum of the form \( \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n. \)

Actually, addition has been defined previously only for a sum of two vectors. To form a sum consisting of \( n \) elements, the sum must be performed two at a time. It follows from the axioms, however, that analogous to the corresponding operations with real numbers, the result is independent of the order of summation. There is thus no ambiguity in the simplified notation.

It is apparent that a linear combination of vectors from a subspace is also in the subspace. Conversely, linear combinations can be used to construct a subspace from an arbitrary subset in a vector space.

**Definition.** Suppose \( S \) is a subset of a vector space \( X. \) The set \( [S] \), called the subspace generated by \( S, \) consists of all vectors in \( X \) which are linear combinations of vectors in \( S. \)

The verification that \( [S] \) is a subspace in \( X \) follows from the obvious fact that a linear combination of linear combinations is also a linear combination.

There is an interesting characterization of the subspace \( [S] \). The set \( S \) is, in general, wholly contained in a number of subspaces. Of these, the subspace \( [S] \) is the smallest in the sense that if \( M \) is a subspace containing \( S, \) then \( M \) contains \( [S]. \) This statement is proved by noting that if the subspace \( M \) contains \( S, \) it must contain all linear combinations from \( S. \)

In three-dimensional space the subspace generated by a two-dimensional circle centered at the origin is a plane. The subspace generated by a plane not passing through the origin is the whole space. A subspace is a generalization of our intuitive notion of a plane or line through the origin. The translation of a subspace, therefore, is a generalization of an arbitrary plane or line.

**Definition.** The translation of a subspace is said to be a linear variety.
A linear variety $V$ can be written as $V = x_0 + M$ where $M$ is a subspace. In this representation the subspace $M$ is unique, but any vector in $V$ can serve as $x_0$. This is illustrated in Figure 2.2.

Given a subset $S$, we can construct the smallest linear variety containing $S$.

![Figure 2.2 A linear variety](image)

**Definition.** Let $S$ be a nonempty subset of a vector space $X$. The linear variety generated by $S$, denoted $v(S)$ is defined as the intersection of all linear varieties in $X$ that contain $S$.

We leave it to the reader to justify the above definition by showing that $v(S)$ is indeed a linear variety.

### 2.4 Convexity and Cones

We come now to the topic that is responsible for a surprising number of the results in this book and which generalizes many of the useful properties of subspaces and linear varieties.

**Definition.** A set $K$ in a linear vector space is said to be convex if, given $x_1, x_2 \in K$, all points of the form $\alpha x_1 + (1 - \alpha)x_2$ with $0 \leq \alpha \leq 1$ are in $K$.

This definition merely says that given two points in a convex set, the line segment between them is also in the set. Examples are shown in Figure 2.3. Note in particular that subspaces and linear varieties are convex. The empty set is considered convex.

---

1 Other names for a linear variety include: flat, affine subspace, and linear manifold. The term linear manifold, although commonly used, is reserved by many authors as an alternative term for subspace.
The following relations for convex sets are elementary but important. Their proofs are left to the reader.

**Proposition 1.** Let $K$ and $G$ be convex sets in a vector space. Then

1. $\alpha K = \{x : x = \alpha k, k \in K\}$ is convex for any scalar $\alpha$.
2. $K + G$ is convex.

We also have the following elementary property.

**Proposition 2.** Let $\mathcal{C}$ be an arbitrary collection of convex sets. Then

$\bigcap_{K \in \mathcal{C}} K$ is convex.

**Proof.** Let $C = \bigcap_{K \in \mathcal{C}} K$. If $C$ is empty, the lemma is trivially proved. Assume that $x_1, x_2 \in C$ and select $\alpha, 0 \leq \alpha \leq 1$. Then $x_1, x_2 \in K$ for all $K \in \mathcal{C}$, and since $K$ is convex, $\alpha x_1 + (1 - \alpha)x_2 \in K$ for all $K \in \mathcal{C}$. Thus $\alpha x_1 + (1 - \alpha)x_2 \in C$ and $C$ is convex.

**Definition.** Let $S$ be an arbitrary set in a linear vector space. The **convex cover** or **convex hull**, denoted $\text{co}(S)$ is the smallest convex set containing $S$. In other words, $\text{co}(S)$ is the intersection of all convex sets containing $S$.

Note that the justification of this definition rests with Proposition 2 since it guarantees the existence of a smallest convex set containing $S$. Some examples of a set and its convex hull are illustrated in Figure 2.4.

**Definition.** A set $C$ in a linear vector space is said to be a **cone with vertex at the origin** if $x \in C$ implies that $\alpha x \in C$ for all $\alpha \geq 0$. 
Several cases are shown in Figure 2.5. A *cone with vertex* $p$ is defined as a translation $p + C$ of a cone $C$ with vertex at the origin. If the vertex of a cone is not explicitly mentioned, it is assumed to be the origin. A *convex cone* is, of course, defined as a set which is both convex and a cone. Of the cones in Figure 2.5, only (b) is a convex cone. Cones again generalize the concepts of subspace and linear variety since both of these are convex cones.

![Figure 2.5 Cones](image)

Convex cones usually arise in connection with the definition of positive vectors in a vector space. For instance, in $n$-dimensional real coordinate space, the set

$$ P = \{ x : x = \{ \xi_1, \xi_2, \ldots, \xi_n \}, \xi_i \geq 0 \ \text{all } i \}, $$

defining the positive orthant, is a convex cone. Likewise, the set of all nonnegative continuous functions is a convex cone in the vector space of continuous functions.

### 2.5 Linear Independence and Dimension

In this section we first introduce the concept of linear independence, which is important for any general study of vector space, and then review the essential distinguishing features of finite-dimensional space: basis and dimension.

**Definition.** A vector $x$ is said to be *linearly dependent* upon a set $S$ of vectors if $x$ can be expressed as a linear combination of vectors from $S$. Equivalently, $x$ is linearly dependent upon $S$ if $x$ is in $[S]$, the subspace generated by $S$. Conversely, the vector $x$ is said to be *linearly independent* of the set $S$ if it is not linearly dependent on $S$; a set of vectors is said to be a *linearly independent set* if each vector in the set is linearly independent of the remainder of the set.
Thus, two vectors are linearly independent if they do not lie on a common line through the origin, three vectors are linearly independent if they do not lie in a plane through the origin, etc. It follows from our definition that the vector \( \mathbf{0} \) is dependent on any given vector \( \mathbf{x} \) since \( \mathbf{0} = 0 \mathbf{x} \). Also, by convention, the set consisting of \( \mathbf{0} \) alone is understood to be a dependent set. On the other hand, a set consisting of a single nonzero vector is an independent set. With these conventions the following major test for linear independence is applicable even to trivial sets consisting of a single vector.

**Theorem 1.** A necessary and sufficient condition for the set of vectors \( x_1, x_2, \ldots, x_n \) to be linearly independent is that the expression \( \sum_{k=1}^{n} a_k x_k = \mathbf{0} \) implies \( a_k = 0 \) for all \( k = 1, 2, 3, \ldots, n \).

**Proof.** To prove the necessity of the condition, assume that \( \sum_{k=1}^{n} a_k x_k = \mathbf{0} \), and that for some index \( r \), \( a_r \neq 0 \). Since the coefficient of \( x_r \) is nonzero, the original relation may be rearranged to produce \( x_r = \sum_{k \neq r} (-a_k/a_r) x_k \) which shows \( x_r \) to be linearly dependent on the remaining vectors.

To prove the sufficiency of the condition, note that linear dependence among the vectors implies that one vector, say \( x_r \), can be expressed as a linear combination of the others, \( x_r = \sum_{k \neq r} a_k x_k \). Rearrangement gives \( \sum_{k \neq r} a_k x_k - x_r = \mathbf{0} \) which is the desired relation.

An important consequence of this theorem is that a vector expressed as a linear combination of linearly independent vectors can be so expressed in only one way.

**Corollary 1.** If \( x_1, x_2, \ldots, x_n \) are linearly independent vectors, and if \( \sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} \beta_k x_k \), then \( a_k = \beta_k \) for all \( k = 1, 2, \ldots, n \).

**Proof.** If \( \sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} \beta_k x_k \), then \( \sum_{k=1}^{n} (a_k - \beta_k) x_k = \mathbf{0} \) and \( a_k - \beta_k = 0 \) according to Theorem 1.

We turn now from the general notion of linear independence to the special topic of finite dimensionality. Consider a set \( S \) of linearly independent vectors in a vector space. These vectors may be used to generate \([S]\), a certain subspace of the vector space. If the subspace \([S]\) is actually the entire vector space, every vector in the space is expressible as a linear combination of vectors from the original set \( S \). Furthermore, according to the corollary to Theorem 1, the expression is unique.

**Definition.** A finite set \( S \) of linearly independent vectors is said to be a basis for the space \( X \) if \( S \) generates \( X \). A vector space having a finite basis is said to be finite dimensional. All other vector spaces are said to be infinite dimensional.
Usually, we characterize a finite-dimensional space by the number of elements in a basis. Thus, a space with a basis consisting of $n$ elements is referred to as $n$-dimensional space. This practice would be undesirable on grounds of ambiguity if the number of elements in a basis for a space were not unique. The following theorem is, for this reason, a fundamental result in the study of finite-dimensional spaces.

**Theorem 2.** Any two bases for a finite-dimensional vector space contain the same number of elements.

*Proof.* Suppose that \(\{x_1, x_2, \ldots, x_n\}\) and \(\{y_1, y_2, \ldots, y_m\}\) are bases for a vector space \(X\). Suppose also that \(m \geq n\). We shall substitute \(y\) vectors for \(x\) vectors one by one in the first basis until all the \(x\) vectors are replaced.

Since the \(x_i\)'s form a basis, the vector \(y_1\) may be expressed as a linear combination of them, say, \(y_1 = \sum_{i=1}^{n} \alpha_i x_i\). Since \(y_1 \neq \theta\), at least one of the scalars \(\alpha_i\) must be nonzero. Rearranging the \(x_i\)'s if necessary, it may be assumed that \(\alpha_1 \neq 0\). Then \(x_1\) may be expressed as a linear combination of \(y_1, x_2, x_3, \ldots, x_n\) by the formula

\[
x_1 = \alpha_1^{-1} y_1 + \sum_{i=2}^{n} \alpha_i^{-1} \alpha_i x_i.
\]

The set \(y_1, x_2, \ldots, x_n\) generates \(X\) since any linear combination of the original \(x_i\)'s becomes an equivalent linear combination of this new set when \(x_1\) is replaced according to the above formula.

Suppose now that \(k - 1\) of the vectors \(x_i\) have been replaced by the first \(k - 1\) \(y_i\)'s. The vector \(y_k\) can be expressed as a linear combination of \(y_1, y_2, \ldots, y_{k-1}, x_k, \ldots, x_n\), say,

\[
y_k = \sum_{i=1}^{k-1} \alpha_i y_i + \sum_{i=k}^{n} \beta_i x_i.
\]

Since the vectors \(y_1, y_2, \ldots, y_k\) are linearly independent, not all the \(\beta_i\)'s can be zero. Rearranging \(x_k, x_{k+1}, \ldots, x_n\) if necessary, it may be assumed that \(\beta_k \neq 0\). Then \(x_k\) can be solved for as a linear combination of \(y_1, y_2, \ldots, y_k, x_{k+1}, \ldots, x_n\), and this new set of vectors generates \(X\).

By induction on \(k\) then, we can replace all \(n\) of the \(x_i\)'s by \(y_i\)'s, forming a generating set at each step. This implies that the independent vectors \(y_1, y_2, \ldots, y_n\) generate \(X\) and hence form a basis for \(X\). Therefore, by the assumption of linear independence of \(\{y_1, y_2, \ldots, y_m\}\), we must have \(n = m\).

Finite-dimensional spaces are somewhat simpler to analyze than infinite-dimensional spaces. Fewer definitions are required, fewer pathological cases arise, and our native intuition is contradicted in fewer instances.
in finite-dimensional spaces. Furthermore, there are theorems and concepts in finite-dimensional space which have no direct counterpart in infinite-dimensional spaces. Occasionally, the availability of these special characteristics of finite dimensionality is essential to obtaining a solution to a particular problem. It is more usual, however, that results first derived for finite-dimensional spaces do have direct analogs in more general spaces. In these cases, verification of the corresponding result in infinite-dimensional space often enhances our understanding by indicating precisely which properties of the space are responsible for the result. We constantly endeavor to stress the similarities between infinite- and finite-dimensional spaces rather than the few minor differences.

**NORMED LINEAR SPACES**

### 2.6 Definition and Examples

The vector spaces of particular interest in both abstract analysis and applications have a good deal more structure than that implied solely by the seven principal axioms. The vector space axioms only describe algebraic properties of the elements of the space: addition, scalar multiplication, and combinations of these. What are missing are the topological concepts such as openness, closure, convergence, and completeness. These concepts can be provided by the introduction of a measure of distance in a space.

**Definition.** A *normed linear vector space* is a vector space $X$ on which there is defined a real-valued function which maps each element $x$ in $X$ into a real number $\|x\|$ called the norm of $x$. The norm satisfies the following axioms:

1. $\|x\| \geq 0$ for all $x \in X$, $\|x\| = 0$ if and only if $x = \theta$.
2. $\|x + y\| \leq \|x\| + \|y\|$ for each $x, y \in X$. (triangle inequality)
3. $\|ax\| = |a| \cdot \|x\|$ for all scalars $a$ and each $x \in X$.

The norm is clearly an abstraction of our usual concept of length. The following useful inequality is a direct consequence of the triangle inequality.

**Lemma 1.** In a normed linear space $\|x\| - \|y\| \leq \|x - y\|$ for any two vectors $x, y$.

**Proof.**

$\|x\| - \|y\| = \|x - y + y\| - \|y\| \leq \|x - y\| + \|y\| - \|y\| = \|x - y\|$. 

By introduction of a suitable norm, many of our earlier examples of vector spaces can be converted to normed spaces.
Example 1. The normed linear space $C[a, b]$ consists of continuous functions on the real interval $[a, b]$ together with the norm $\|x\| = \max_{a \leq t \leq b} |x(t)|$. This space was considered as a vector space in Section 2.2. We now verify that the proposed norm satisfies the three required axioms. Obviously, $\|x\| \geq 0$ and is zero only for the function which is identically zero. The triangle inequality follows from the relation

$$\max |x(t) + y(t)| \leq \max [|x(t)| + |y(t)|] \leq \max |x(t)| + \max |y(t)|.$$ 

Finally, the third axiom follows from the relation

$$\max |\alpha x(t)| = |\alpha| \max |x(t)| = |\alpha| \max |x(t)|.$$ 

Example 2. The normed linear space $D[a, b]$ consists of all functions on the interval $[a, b]$ which are continuous and have continuous derivatives on $[a, b]$. The norm on the space $D[a, b]$ is defined as

$$\|x\| = \max_{a \leq t \leq b} |x(t)| + \max_{a \leq t \leq b} |\dot{x}(t)|.$$ 

We leave it to the reader to verify that $D[a, b]$ is a normed linear space.

Example 3. The space of finitely nonzero sequences together with the norm equal to the sum of the absolute values of the nonzero components is a normed linear space. Thus the element $x = \{\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots\}$ has its norm defined as $\|x\| = \sum_{i=1}^{n} |\xi_i|$. We may easily verify the three required properties by inspection.

Example 6. The space of continuous functions on the interval $[a, b]$ becomes a normed space with the norm of a function $x$ defined as

$$\|x\| = \int_a^b |x(t)| \, dt.$$ 

This is a different normed space than $C[a, b]$.

Example 5. Euclidean $n$-space, denoted $E^n$, consists of $n$-tuples with the norm of an element $x = \{\xi_1, \xi_2, \ldots, \xi_n\}$ defined as $\|x\| = (\sum_{i=1}^{n} |\xi_i|^2)^{1/2}$. This definition obviously satisfies the first and third axioms for norms. The triangle inequality for this norm is a well-known result from finite-dimensional vector spaces and is a special case of the Minkowski inequality discussed in Section 2.10. The space $E^n$ can be chosen as a real or complex space by considering real or complex $n$-tuples. We employ the same notation $E^n$ for both because it is generally apparent from context which is meant.

Example 6. We consider now the space $BV[a, b]$ consisting of functions of bounded variation on the interval $[a, b]$. By a partition of the interval $[a, b]$, we mean a finite set of points $t_i \in [a, b]$, $i = 0, 1, 2, \ldots, n$, such that $a = t_0 < t_1 < t_2 < \cdots < t_n = b$. A function $x$ defined on $[a, b]$ is said to...
be of bounded variation if there is a constant \( K \) so that for any partition of \([a, b]\)

\[
\sum_{i=1}^{n}|x(t_i) - x(t_{i-1})| \leq K.
\]

The total variation of \( x \), denoted \( \text{T.V.}(x) \), is then defined as

\[
\text{T.V.}(x) = \sup \sum_{i=1}^{n}|x(t_i) - x(t_{i-1})|
\]

where the supremum is taken with respect to all partitions of \([a, b]\). A convenient and suggestive notation for the total variation is

\[
\text{T.V.}(x) = \int_a^b |dx(t)|.
\]

The total variation of a constant function is zero and the total variation of a monotonic function is the absolute value of the difference between the function values at the end points \( a \) and \( b \).

The space \( BV[a, b] \) is defined as the space of all functions of bounded variation on \([a, b]\) together with the norm defined as

\[
\|x\| = |x(a)| + \text{T.V.}(x).
\]

2.7 Open and Closed Sets

We come now to the concepts that are fundamental to the study of topological properties.

**Definition.** Let \( P \) be a subset of a normed space \( X \). The point \( p \in P \) is said to be an interior point of \( P \) if there is an \( \varepsilon > 0 \) such that all vectors \( x \) satisfying \( \|x - p\| < \varepsilon \) are also members of \( P \). The collection of all interior points of \( P \) is called the interior of \( P \) and is denoted \( \overset{\circ}{P} \).

We introduce the notation \( S(x, \varepsilon) \) for the (open) sphere centered at \( x \) with radius \( \varepsilon \); that is, \( S(x, \varepsilon) = \{y : \|x - y\| < \varepsilon\} \). Thus, according to the above definition, a point \( x \) is an interior point of \( P \) if there is a sphere \( S(x, \varepsilon) \) centered at \( x \) and contained in \( P \). A set may have an empty interior as, for example, a set consisting of a single point or a line in \( E^2 \).

**Definition.** A set \( P \) is said to be open if \( P = \overset{\circ}{P} \).

The empty set is open since its interior is also empty. The entire space is an open set. The unit sphere consisting of all vectors \( x \) with \( \|x\| < 1 \) is an open set. We leave it to the reader to verify that \( \overset{\circ}{P} \) is open.
\textbf{Definition.} A point \( x \in X \) is said to be a \textit{closure point} of a set \( P \) if, given \( \varepsilon > 0 \), there is a point \( p \in P \) satisfying \( \| x - p \| < \varepsilon \). The collection of all closure points of \( P \) is called the \textit{closure} of \( P \) and is denoted \( \overline{P} \).

In other words, a point \( x \) is a closure point of \( P \) if every sphere centered at \( x \) contains a point of \( P \). It is clear that \( P \subseteq \overline{P} \).

\textbf{Definition.} A set \( P \) is said to be \textit{closed} if \( P = \overline{P} \).

The empty set and the whole space \( X \) are closed as well as open. The unit sphere consisting of all points \( x \) with \( \| x \| \leq 1 \) is a closed set. A single point is a closed set. It is clear that \( \overline{\{ x \}} = \overline{P} \).

\textbf{Proposition 1.} The complement of an open set is closed and the complement of a closed set is open.

\textit{Proof.} Let \( P \) be an open set and \( \overline{P} = \{ x : x \notin P \} \) its complement. A point \( x \) in \( P \) is not a closure point of \( \overline{P} \) since there is a sphere about \( x \) disjoint from \( \overline{P} \). Thus \( \overline{P} \) contains all its closure points and is therefore closed.

Let \( S \) be a closed set. If \( x \in \overline{S} \), then \( x \) is not a closure point of \( S \) and, hence, there is a sphere about \( x \) which is disjoint from \( \overline{S} \). Therefore \( x \) is an interior point of \( \overline{S} \). We conclude that \( \overline{S} \) is open. \( \blacksquare \)

The proofs of the following two complementary results are left to the reader.

\textbf{Proposition 2.} The intersection of a finite number of open sets is open; the union of an arbitrary collection of open sets is open.

\textbf{Proposition 3.} The union of a finite number of closed sets is closed; the intersection of an arbitrary collection of closed sets is closed.

We now have two topological operations, taking closures and taking interiors, that can be applied to sets in normed space. It is natural to investigate the effect of these operations on convexity, the fundamental algebraic concept of vector space.

\textbf{Proposition 4.} Let \( C \) be a convex set in a normed space. Then \( \overline{C} \) and \( \overline{C} \) are convex.

\textit{Proof.} If \( \overline{C} \) is empty, it is convex. Suppose \( x_0, y_0 \) are points in \( \overline{C} \). Fix \( \alpha \), \( 0 < \alpha < 1 \). We must show that \( z_0 = \alpha x_0 + (1 - \alpha)y_0 \in \overline{C} \). Given \( \varepsilon > 0 \), let \( x, y \) be selected from \( C \) such that \( \| x - x_0 \| < \varepsilon \), \( \| y - y_0 \| < \varepsilon \). Then \( \| \alpha x + (1 - \alpha)y - \alpha x_0 - (1 - \alpha)y_0 \| \leq \varepsilon \) and, hence, \( z_0 \) is within a distance \( \varepsilon \) of \( z = \alpha x + (1 - \alpha)y \) which is in \( C \). Since \( \varepsilon \) is arbitrary, it follows that \( z_0 \) is a closure point of \( C \).
If $\mathcal{C}$ is empty, it is convex. Suppose $x_0, y_0 \in \mathcal{C}$ and fix $\alpha, 0 < \alpha < 1$. We must show that $z_0 = \alpha x_0 + (1 - \alpha) y_0 \in \mathcal{C}$. Since $x_0, y_0 \in \mathcal{C}$, there is an $\varepsilon > 0$ such that the open spheres $S(x_0, \varepsilon), S(y_0, \varepsilon)$ are contained in $\mathcal{C}$. It follows that all points of the form $z_0 + w$ with $\|w\| < \varepsilon$ are in $\mathcal{C}$ since $z_0 + w = \alpha(x_0 + w) + (1 - \alpha)(y_0 + w)$. Thus, $z_0$ is an interior point of $\mathcal{C}$.

Likewise, it can be shown that taking closure preserves subspaces, linear varieties, and cones.

Finally, we remark that all of the topological concepts discussed above can be defined relative to a given linear variety. Suppose that $P$ is a set contained in a linear variety $V$. We say that $p \in P$ is an interior point of $P$ relative to $V$ if there is an $\varepsilon > 0$ such that all vectors $x \in V$ satisfying $\|x - p\| < \varepsilon$ are also members of $P$. The set $P$ is said to be open relative to $V$ if every point in $P$ is an interior point of $P$ relative to $V$.

In case $V$ is taken as the closed linear variety generated by $P$, i.e., the intersection of all closed linear varieties containing $P$, then $x$ is simply referred to as a relative interior point of $P$ if it is an interior point of $P$ relative to the variety $V$. Similar meaning is given to relatively closed, etc.

2.8 Convergence

In order to prove the existence of a vector satisfying a desired property, it is common to establish an appropriate sequence of vectors converging to a limit. In many cases the limit can be shown to satisfy the required property. It is for this reason that the concept of convergence plays an important role in analysis.

**Definition.** In a normed linear space an infinite sequence of vectors $\{x_n\}$ is said to converge to a vector $x$ if the sequence $\{\|x - x_n\|\}$ of real numbers converges to zero. In this case, we write $x_n \to x$.

If $x_n \to x$, it follows that $\|x_n\| \to \|x\|$ because, according to Lemma 1, Section 2.6, we have both $\|x_n\| - \|x\| \leq \|x_n - x\|$ and $\|x\| - \|x_n\| \leq \|x_n - x\|$ which implies that $\|x_n\| - \|x\| \leq \|x_n - x\| \to 0$.

In the space $\mathbb{R}^n$ of $n$-tuples, a sequence converges if and only if each component converges; however, in other spaces convergence is not always easy to characterize. In the space of finitely nonzero sequences, define the vectors $e_i = \{0, 0, \ldots, 1, 0, \ldots\}$, the $i$-th vector having each of its components zero except the $i$-th which is 1. The sequence of vectors $\{e_i\}$ (which is now a sequence of sequences) converges to zero componentwise, but the sequence does not converge to the null vector since $\|e_i\| = 1$ for all $i$.

An important observation is stated in the following proposition.
Proposition 1. If a sequence converges, its limit is unique.

Proof. Suppose \( x_n \to x \) and \( x_n \to y \). Then

\[
\|x - y\| = \|x - x_n + x_n - y\| \leq \|x - x_n\| + \|x_n - y\| \to 0.
\]

Thus, \( x = y \).

Another way to state the definition of convergence is in terms of spheres. A sequence \( \{x_n\} \) converges to \( x \) if and only if given \( \varepsilon > 0 \); the sphere \( S(x, \varepsilon) \) contains \( x_n \) for all \( n \) greater than some number \( N \).

The definition of convergence can be used to characterize closed sets and provides a useful alternative to the original definition of closed sets.

Proposition 2. A set \( F \) is closed if and only if every convergent sequence with elements in \( F \) has its limit in \( F \).

Proof. The limit of a sequence from \( F \) is obviously a closure point of \( F \) and, therefore, must be contained in \( F \) if \( F \) is closed. Suppose now that \( F \) is not closed. Then there is a closure point \( x \) of \( F \) that is not in \( F \). In each of the spheres \( S(x, 1/n) \) we may select a point \( x_n \in F \) since \( x \) is a closure point. The sequence \( \{x_n\} \) generated in this way converges to \( x \not\in F \).

2.9 Transformations and Continuity

The objects that make the study of linear spaces interesting and useful are transformations.

Definition. Let \( X \) and \( Y \) be linear vector spaces and let \( D \) be a subset of \( X \). A rule which associates with every element \( x \in D \) an element \( y \in Y \) is said to be a transformation from \( X \) to \( Y \) with domain \( D \). If \( y \) corresponds to \( x \) under \( T \), we write \( y = T(x) \).

Transformations on vector spaces become increasingly more important as we progress through this book; they are treated in some detail beginning with Chapter 6. It is the purpose of this section to introduce some common terminology that is convenient for describing the simple transformations encountered in the early chapters.

If a specific domain is not explicitly mentioned when discussing a transformation on a vector space \( X \), it is understood that the domain is \( X \) itself. If for every \( y \in Y \) there is at most one \( x \in D \) for which \( T(x) = y \), the transformation \( T \) is said to be one-to-one. If for every \( y \in Y \) there is at least one \( x \in D \) for which \( T(x) = y \), \( T \) is said to be onto or, more precisely, to map \( D \) onto \( Y \). This terminology, as the notion of a transformation itself, is of course an extension of the familiar notion of ordinary functions. A transformation is simply a function defined on one vector space \( X \) while
taking values in another vector space $Y$. A special case of this situation is that in which the space $Y$ is taken to be the real line.

**Definition.** A transformation from a vector space $X$ into the space of real (or complex) scalars is said to be a functional on $X$.

In order to distinguish functionals from more general transformations, they are usually denoted by lower case letters such as $f$ and $g$. Hence, $f(x)$ denotes the scalar that $f$ associates with the vector $x \in X$.

On a normed space, $f(x) = \|x\|$ is an example of a functional. On the space $C[0,1]$, examples of functionals are $f_1(x) = x(\frac{1}{2})$, $f_2(x) = \int_0^1 x(t) \, dt$, $f_3(x) = \max_{0 \leq t \leq 1} x^4(t)$, etc. Real-valued functionals are of direct interest to optimization theory, since optimization consists of selecting a vector in minimize (or maximize) a prescribed functional.

**Definition.** A transformation $T$ mapping a vector space $X$ into a vector space $Y$ is said to be linear if for every $x_1, x_2 \in X$ and all scalars $\alpha_1, \alpha_2$ we have $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$.

The most familiar example of a linear transformation is supplied by a rectangular $m \times n$ matrix mapping elements of $\mathbb{R}^n$ into $\mathbb{R}^m$. An example of a linear transformation mapping $X = C[a, b]$ into $X$ is the integral operator $T(x) = \int_a^b k(t, \tau) x(\tau) \, d\tau$ where $k(t, \tau)$ is a function continuous on the square $a \leq t \leq b$; $a \leq \tau \leq b$.

Up to this point we have considered transformations mapping one abstract space into another. If these spaces happen to be normed, it is possible to define the notion of continuity.

**Definition.** A transformation $T$ mapping a normed space $X$ into a normed space $Y$ is continuous at $x_0 \in X$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|x - x_0\| < \delta$ implies that $\|T(x) - T(x_0)\| < \varepsilon$.

Note that continuity depends on the norm in both the spaces $X$ and $Y$. If $T$ is continuous at each point $x_0 \in X$, we say that $T$ is continuous everywhere or, more simply, that $T$ is continuous.

The following characterization of continuity is useful in many proofs involving continuous transformations.

**Proposition 1.** A transformation $T$ mapping a normed space $X$ into a normed space $Y$ is continuous at the point $x_0 \in X$ if and only if $x_n \to x_0$ implies $T(x_n) \to T(x_0)$.

**Proof.** The "if" portion of the statement is obvious; thus we need only prove the "only if" portion. Let $\{x_n\}$ be a sequence such that $x_n \to x_0$, $T(x_n) \to T(x_0)$. Then, for some $\varepsilon > 0$ and every $N$ there is an $n > N$ such
that \( \|T(x_n) - T(x_0)\| \geq \varepsilon \). Since \( x_n \to x_0 \), this implies that for every \( \delta > 0 \) there is a point \( x_n \) with \( \|x_n - x_0\| < \delta \) and \( \|T(x_n) - T(x)\| > \varepsilon \). This proves the " only if" portion by contraposition.

*2.10 The \( l_p \) and \( L_p \) Spaces

In this section we discuss some classical normed spaces that are useful throughout the book.

**Definition.** Let \( p \) be a real number \( 1 \leq p < \infty \). The space \( \ell_p \) consists of all sequences of scalars \( \{ \xi_1, \xi_2, \ldots \} \) for which
\[
\sum_{i=1}^{\infty} |\xi_i|^p < \infty.
\]
The norm of an element \( x = \{ \xi_i \} \) in \( l_p \) is defined as
\[
\|x\|_p = \left( \sum_{i=1}^{\infty} |\xi_i|^p \right)^{1/p}.
\]
The space \( l_\infty \) consists of bounded sequences. The norm of an element \( x = \{ \xi_i \} \) in \( l_\infty \) is defined as
\[
\|x\|_\infty = \sup_i |\xi_i|.
\]

It is obvious that the norm on \( l_p \) satisfies \( \|ax\| = |a| \|x\| \) and that \( \|x\| > 0 \) for each \( x \neq \theta \). In this section, we establish two inequalities concerning the \( l_p \) norms, the second of which gives the triangle inequality for these \( l_p \) norms. Therefore, the \( l_p \) norm indeed satisfies the three axioms required of a general norm. Incidentally, it follows from these properties of the norm that \( l_p \) is in fact a linear vector space because, if \( x = \{ \xi_i \} \), \( y = \{ \eta_i \} \) are vectors in \( l_p \), then for any scalars \( \alpha, \beta \), we have \( \|\alpha x + \beta y\| \leq |\alpha| \|x\| + |\beta| \|y\| < \infty \) so that \( \alpha x + \beta y \) is a vector in \( l_p \). Since \( l_p \) is a vector space and the norm satisfies the three required axioms, we may justifiably refer to \( l_p \) as a normed linear vector space.

The following two theorems, although of fundamental importance for a study of the \( l_p \) spaces, are somewhat tangential to our main purpose. The reader will lose little by simply scanning the proofs.

**Theorem 1.** (The H"older Inequality) If \( p \) and \( q \) are positive numbers \( 1 \leq p \leq \infty, 1 \leq q \leq \infty \), such that \( 1/p + 1/q = 1 \) and if \( x = \{ \xi_1, \xi_2, \ldots \} \in \ell_p \)
\[
y = \{ \eta_1, \eta_2, \ldots \} \in \ell_q,
\]
then
\[
\sum_{i=1}^{\infty} |\xi_i \eta_i| \leq \|x\|_p \cdot \|y\|_q.
\]
Equality holds if and only if
\[
\left( \frac{||x||_p}{||y||_q} \right)^{1/q} = \left( \frac{||y||_q}{||x||_p} \right)^{1/p}
\]
for each \(i\).

Proof. The cases \(p = 1, \infty; q = \infty, 1\) are straightforward and are left to the reader. Therefore, it is assumed that \(1 < p < \infty, 1 < q < \infty\). We first prove the auxiliary inequality: For \(a \geq 0, b \geq 0, \text{ and } 0 < \lambda < 1\), we have
\[
a^{1-\lambda}b^{\lambda} \leq \lambda a + (1 - \lambda)b
\]
with equality if and only if \(a = b\).

For this purpose, consider the function
\[
f(t) = t^\lambda - \lambda t + \lambda - 1
\]
defined for \(t \geq 0\). Then \(f'(t) = \lambda(t^{\lambda-1} - 1)\). Since \(0 < \lambda < 1\), we have \(f'(t) > 0\) for \(0 < t < 1\) and \(f'(t) < 0\) for \(t > 1\). It follows that for \(t \geq 0\), \(f(t) \leq f(1) = 0\) with equality only for \(t = 1\). Hence
\[
t^\lambda \leq \lambda t + 1 - \lambda
\]
with equality only for \(t = 1\). If \(b \neq 0\), the substitution \(t = a/b\) gives the desired inequality, while for \(b = 0\) the inequality is trivial.

Applying this inequality to the numbers
\[
a = \left( \frac{||x||_p}{||y||_q} \right)^{1/p}, \quad b = \left( \frac{||y||_q}{||x||_p} \right)^{1/q}
\]
we obtain for each \(i\)
\[
\frac{||x_i \eta_i||}{||x||_p ||y||_q} \leq \frac{1}{p} \left( \frac{||x_i||}{||x||_p} \right)^p + \frac{1}{q} \left( \frac{||\eta_i||}{||y||_q} \right)^q.
\]
Summing this inequality over \(i\), we obtain the Hölder inequality
\[
\sum_{i=1}^{\infty} \frac{||x_i \eta_i||}{||x||_p ||y||_q} \leq \frac{1}{p} + \frac{1}{q} = 1.
\]
The conditions for equality follow directly from the required condition \(a = b\) in the auxiliary inequality.

The special case \(p = 2, q = 2\) is of major importance. In this case, Hölder's inequality becomes the well-known Cauchy-Schwarz inequality for sequences:
\[
\sum_{i=1}^{\infty} \frac{||x_i \eta_i||}{||x||_p ||y||_q} \leq \left( \sum_{i=1}^{\infty} ||x_i||^2 \right)^{1/2} \left( \sum_{i=1}^{\infty} ||\eta_i||^2 \right)^{1/2},
\]
Using the Hölder inequality, we can establish the triangle inequality for $l_p$ norms.

**Theorem 2. (The Minkowski Inequality)** If $x$ and $y$ are in $l_p$, $1 \leq p \leq \infty$, then so is $x + y$, and $\|x + y\|_p \leq \|x\|_p + \|y\|_p$. For $1 < p < \infty$, equality holds if and only if $k_1x = k_2y$ for some positive constants $k_1$ and $k_2$.

**Proof.** The cases $p = 1$ and $p = \infty$ are straightforward. Therefore, it is assumed that $1 < p < \infty$. We first consider finite sums. Clearly we may write

$$\sum_{i=1}^{n} |\xi_i + \eta_i|^p \leq \sum_{i=1}^{n} |\xi_i| |\eta_i|^p.$$  

Applying Hölder's inequality to each summation on the right, we obtain

$$\sum_{i=1}^{n} |\xi_i + \eta_i|^p \leq \left( \left( \sum_{i=1}^{n} |\xi_i|^p \right)^{1/p} + \left( \sum_{i=1}^{n} |\eta_i|^p \right)^{1/p} \right)^{1/q} \left( \sum_{i=1}^{n} |\xi_i|^q \right)^{1/q}.$$  

Dividing both sides of this inequality by $(\sum_{i=1}^{n} |\xi_i + \eta_i|^p)^{1/p}$ and taking account of $1 - 1/q = 1/p$, we find

$$\left( \sum_{i=1}^{n} |\xi_i + \eta_i|^p \right)^{1/p} \leq \left( \sum_{i=1}^{n} |\xi_i|^p \right)^{1/p} + \left( \sum_{i=1}^{n} |\eta_i|^p \right)^{1/p}.$$  

Letting $n \to \infty$ on the right side of this inequality can only increase its value, so

$$\left( \sum_{i=1}^{n} |\xi_i + \eta_i|^p \right)^{1/p} \leq \|x\|_p + \|y\|_p$$

for each $n$. Therefore, letting $n \to \infty$ on the left side produces $\|x + y\|_p \leq \|x\|_p + \|y\|_p$.

The conditions for equality follow from the conditions for equality in the Hölder inequality and are left to the reader. 

The $L_p$ and $R_p$ spaces are defined analogously to the $l_p$ spaces. For $p \geq 1$, the space $L_p[a, b]$ consists of those real-valued measurable functions $x$ on the interval $[a, b]$ for which $|x(t)|^p$ is Lebesgue integrable. The norm on this space is defined as

$$\|x\|_p = \left( \int_a^b |x(t)|^p \, dt \right)^{1/p}.$$  

Unfortunately, on this space $\|x\|_p = 0$ does not necessarily imply $x = \theta$ since $x$ may be nonzero on a set of measure zero (such as a set consisting of a countable number of points). If, however, we do not distinguish
between functions that are equal almost everywhere, then $L_p[a, b]$ is a normed linear space.

The space $R_p[a, b]$ is defined in an analogous manner but with attention restricted to functions $x$ for which $|x(t)|$ is Riemann integrable. Since all functions encountered in applications in this book are Riemann integrable, we have little reason, at this point in the development, to prefer one of these spaces over the other. Readers unfamiliar with Lebesgue measure and integration theory lose no essential insight by considering the $R_p$ spaces only. In the next section, however, when considering the notion of completeness, we find reason to prefer the $L_p$ spaces over the $R_p$ spaces.

The space $L_\infty[a, b]$ is the function space analog of $l_\infty$. It is defined as the space of all Lebesgue measurable functions on $[a, b]$ which are bounded, except possibly on a set of measure zero. Again, in this space, two functions differing only on a set of measure zero are regarded as equivalent.

Roughly speaking, the norm of an element $x$ in $L_\infty[a, b]$ is defined as $\sup_{a \leq t \leq b} |x(t)|$. This quantity is ambiguous, however, since an element $x$ does not correspond uniquely to any given function but to a whole class of functions differing on a set of measure zero. The value $\sup_{a \leq t \leq b} |x(t)|$ is different for the different functions equivalent to $x$. The norm of a function in $L_\infty[a, b]$ is therefore defined as

$$\|x\|_\infty = \text{essential supremum of } |x(t)| = \inf_{y(t) = x(t) \text{ a.e.}} \sup_{a \leq t \leq b} |y(t)|$$

For brevity, we write $\|x\|_\infty = \text{ess sup } |x(t)|$.

**Example 1.** Consider the function

$$x(t) = \begin{cases} 1 - t^2 & t \in [-1, 1], \\ 2 & t = 0 \end{cases}$$

shown in Figure 2.6. The supremum of this function is equal to 2 but the essential supremum is 1.

There are Hölder and Minkowski inequalities for the $L_p$ spaces analogous to the corresponding results for the $l_p$ spaces.

**Theorem 3.** (The Hölder Inequality) If $x \in L_p[a, b]$, $y \in L_q[a, b]$, $1/p + 1/q = 1$, $p, q > 1$, then

$$\int_a^b |x(t)y(t)| \, dt \leq \|x\|_p \|y\|_q.$$
Equality holds if and only if:

\[
\left( \frac{|x(t)|}{\|x\|_p} \right)^p = \left( \frac{|y(t)|}{\|y\|_q} \right)^q
\]

almost everywhere on \([a, b]\).

**Theorem 4.** (The Minkowski Inequality) If \(x\) and \(y\) are in \(L_p[a, b]\), then so is \(x + y\), and \(\|x + y\|_p \leq \|x\|_p + \|y\|_p\).

The proofs of these inequalities are similar to those for the \(l_p\) spaces and are omitted here.

![Figure 2.6 The function for Example 1](image)

### 2.11 Banach Spaces

**Definition.** A sequence \(\{x_n\}\) in a normed space is said to be a **Cauchy sequence** if \(\|x_n - x_m\| \to 0\) as \(n, m \to \infty\); i.e., given \(\varepsilon > 0\), there is an integer \(N\) such that \(\|x_n - x_m\| < \varepsilon\) for all \(n, m > N\).

In a normed space, every convergent sequence is a Cauchy sequence since, if \(x_n \to x\), then

\[
\|x_n - x_m\| = \|x_n - x + x - x_m\| \leq \|x_n - x\| + \|x - x_m\| \to 0.
\]

In general, however, a Cauchy sequence may not be convergent.

Normed spaces in which every Cauchy sequence is convergent are of particular interest in analysis; in such spaces it is possible to identify convergent sequences without explicitly identifying their limits. A space in which every Cauchy sequence has a limit (and is therefore convergent) is said to be complete.
Definition. A normed linear vector space $X$ is complete if every Cauchy sequence from $X$ has a limit in $X$. A complete normed linear vector space is called a Banach space.

We frequently take great care to formulate problems arising in applications as equivalent problems in Banach space rather than as problems in other, possibly incomplete, spaces. The principal advantage of Banach space in optimization problems is that when seeking an optimal vector maximizing a given objective, we often construct a sequence of vectors, each member of which is superior to the preceding members; the desired optimal vector is then the limit of the sequence. In order that the scheme be effective, there must be available a test for convergence which can be applied when the limit is unknown. The Cauchy criterion for convergence meets this requirement provided the underlying space is complete.

We now consider examples of incomplete normed spaces.

Example 1. Let $X$ be the space of continuous functions on $[0, 1]$ with norm defined by $\|x\| = \int_0^1 |x(t)| \, dt$. One may readily verify that $X$ is a normed linear space. Note that the space is not the space $C[0, 1]$ since the norm is different. We show that $X$ is incomplete. Define a sequence of elements in $X$ by the equation

$$x_n(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq \frac{1}{2} - \frac{1}{n} \\ nt - \frac{n}{2} + 1 & \text{for } \frac{1}{2} - \frac{1}{n} \leq t \leq \frac{1}{2} \\ 1 & \text{for } t \geq \frac{1}{2} \end{cases}$$

This sequence of functions is illustrated in Figure 2.7. Each member of the sequence is a continuous function and thus a member of the space $X$. The sequence is Cauchy since, as is easily verified, $\|x_n - x_m\| = \frac{1}{n} |1/n - 1/m| \to 0$. It is obvious, however, that there is no continuous function to which the sequence converges.

Example 2. Let $X$ be the vector space of finitely nonzero sequences $x = \{\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots\}$. Define the norm of an element $x$ as $\|x\| = \max_i |\xi_i|$. Define a sequence of elements in $X$ by

$$x_n = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-1}, 0, 0, \ldots \right\}.$$
Each \( x_n \) is in \( X \) and has its norm \textit{equal to} unity. The sequence \( \{x_n\} \) is a Cauchy sequence since, as is easily verified,
\[
\|x_n - x_m\| = \max \{1/n, 1/m\} \to 0.
\]
It is obvious, however, that there is no element of \( X \) (with a finite number of nonzero components) to which this sequence converges.

![Figure 2.7 Sequence for Example 1](image)

The space \( E^1 \), the real line, is the fundamental example of a complete space. It is assumed here that the completeness of \( E^1 \), a major topic in elementary analysis, is well known. The completeness of \( E^1 \) is used to establish the completeness of various other normed spaces.

We establish the completeness of several important spaces that are used throughout this book. For this purpose the following lemma is useful.

\textbf{Lemma 1.} \textit{A Cauchy sequence is bounded.}

\textit{Proof.} Let \( \{x_n\} \) be a Cauchy sequence and let \( N \) be an integer such that \( \|x_n - x_N\| < 1 \) for \( n > N \). For \( n > N \), we have
\[
\|x_n\| = \|x_n - x_N + x_N\| \leq \|x_N\| + \|x_n - x_N\| < \|x_N\| + 1.
\]

\textit{Example 3.} \( C[0, 1] \) is a Banach space. We have previously considered this space as a normed space. To prove that \( C[0, 1] \) is complete, it is only necessary to show that every Cauchy sequence in \( C[0, 1] \) has a limit.

Suppose \( \{x_n\} \) is a Cauchy sequence in \( C[0, 1] \). For each fixed \( t \in [0, 1] \),
\[
|x_n(t) - x_m(t)| \leq \|x_n - x_m\| \to 0,
\]
so \( \{x_n(t)\} \) is a Cauchy sequence of real numbers. Since the set of real numbers is complete, there is a real number \( x(t) \) to which the sequence converges; \( x_n(t) \to x(t) \).
Therefore, the functions \( x_n \) converge pointwise to the function \( x \).

We prove next that this pointwise convergence is actually uniform in \( t \in [0, 1] \), i.e., given \( \varepsilon > 0 \), there is an \( N \) such that \( |x_n(t) - x(t)| < \varepsilon \) for all
Given $t \in [0, 1]$ and $n \geq N$. Given $\varepsilon > 0$, choose $N$ such that $\|x_n - x_m\| < \varepsilon/2$ for $n, m > N$. Then for $n > N$

$$|x_n(t) - x(t)| \leq |x_n(t) - x_m(t)| + |x_m(t) - x(t)|$$

$$\leq \|x_n - x_m\| + |x_m(t) - x(t)|.$$  

By choosing $m$ sufficiently large (which may depend on $t$), each term on the right can be made smaller than $\varepsilon/2$ so $|x_n(t) - x(t)| < \varepsilon$ for $n > N$.

We must still prove that the function $x$ is continuous and that the sequence \( \{x_n\} \) converges to $x$ in the norm of $C[0, 1]$. To prove the continuity of $x$, fix $\varepsilon > 0$. For every $\delta$, $t$, and $n$,

$$|x(t + \delta) - x(t)| \leq |x(t + \delta) - x_n(t + \delta)| + |x_n(t + \delta) - x_n(t)| + |x_n(t) - x(t)|.$$  

Since \( \{x_n\} \) converges uniformly to $x$, $n$ may be chosen to make both the first and the last terms less than $\varepsilon/3$ for all $\delta$. Since $x_n$ is continuous, $\delta$ may be chosen to make the second term less than $\varepsilon/3$. Therefore, $x$ is continuous. The convergence of $x_n$ to $x$ in the $C[0, 1]$ norm is a direct consequence of the uniform convergence.

It is instructive to reconcile the completeness of $C[0, 1]$ with Example 1 in which a sequence of continuous functions was shown to be Cauchy but nonconvergent with respect to the integral norm. The difference is that, with respect to the $C[0, 1]$ norm, the sequence defined in Example 1 is not Cauchy. The reader may find it useful to compare these two cases in detail.

**Example 4.** $l_p$, $1 \leq p \leq \infty$ is a Banach space. Assume first that $1 \leq p < \infty$. Let \( \{x_n\} \) be a Cauchy sequence in $l_p$. Then, if $x_n = \{\xi^n_1, \xi^n_2, \ldots \}$, we have

$$|\xi^n_k - \xi^m_k| \leq \left( \sum_{i=1}^{\infty} |\xi^n_i - \xi^m_i|^p \right)^{1/p} \rightarrow 0$$

Hence, for each $k$, the sequence \( \{\xi^n_k\} \) is a Cauchy sequence of real numbers and, therefore, converges to a limit $\bar{\xi}_k$. We show now that the element $x = \{\xi_1, \xi_2, \ldots \}$ is a vector in $l_p$. According to Lemma 1, there is a constant $M$ which bounds the sequence \( \{x_n\} \). Hence, for all $n$ and $k$

$$\sum_{i=1}^{k} |\xi^n_i|^p \leq \|x_n\|^p \leq MP.$$  

Since the left member of the inequality is a finite sum, the inequality remains valid as $n \rightarrow \infty$; therefore,

$$\sum_{i=1}^{k} |\xi_i|^p \leq MP.$$
§2.11 BANACH SPACES

Since this inequality holds uniformly in \( k \), it follows that

\[
\sum_{i=1}^{\infty} |\xi_i|^p \leq M^p
\]

and \( x \in l_p \); the norm of \( x \) is bounded by \( \|x\| \leq M \).

It remains to be shown that the sequence \( \{x_n\} \) converges to \( x \) in the \( l_p \) norm. Given \( \varepsilon > 0 \), there is an \( N \) such that

\[
\sum_{i=1}^{k} |\xi^n_i - \xi^m_i|^p \leq \|x_n - x_m\|^p \leq \varepsilon
\]

for \( n, m > N \) and for each \( k \). Letting \( m \to \infty \) in the finite sum on the left, we conclude that

\[
\sum_{i=1}^{k} |\xi^n_i - \xi_i|^p \leq \varepsilon
\]

for \( n > N \). Now letting \( k \to \infty \), we deduce that \( \|x_n - x\|^p \leq \varepsilon \) for \( n > N \) and, therefore, \( x_n \to x \). This completes the proof for \( 1 \leq p < \infty \).

Now let \( \{x_n\} \) be a Cauchy sequence in \( l_\infty \). Then \( |\xi^n_k - \xi^m_k| \leq \|x_n - x_m\| \to 0 \). Hence, for each \( k \) there is a real number \( \xi_k \) such that \( \xi^n_k \to \xi_k \). Furthermore, this convergence is uniform in \( k \). Let \( x = \{\xi_1, \xi_2, \ldots\} \). Since \( \{x_n\} \) is Cauchy, there is a constant \( M \) such that for all \( n \), \( \|x_n\| \leq M \). Therefore, for each \( k \) and each \( n \), we have \( |\xi^n_k| \leq \|x_n\| \leq M \) from which it follows that \( x \in l_\infty \) and \( \|x\| \leq M \).

The convergence of \( x_n \) to \( x \) follows directly from the uniform convergence of \( \xi^n_k \to \xi_k \).

**Example 5.** \( L_p[0, 1], 1 \leq p \leq \infty \) is a Banach space. We do not prove the completeness of the \( L_p \) spaces because the proof requires a fairly thorough familiarity with Lebesgue integration theory. Consider instead the space \( R_p \) consisting of all functions \( x \) on \([0, 1]\) for which \( |x|^p \) is Riemann integrable with norm defined as

\[
\|x\| = \left\{ \int_0^1 |x(t)|^p \, dt \right\}^{1/p}.
\]

The normed space \( R_p \) is incomplete. It may be completed by adjoining to it certain additional functions derived from Cauchy sequences in \( R_p \). In this way, \( R_p \) is imbedded in a larger normed space which is complete. The smallest complete space containing \( R_p \) is \( L_p \). A general method for completing a normed space is discussed in Problem 15.

**Example 6.** Given two normed spaces \( X, Y \), we consider the product space \( X \times Y \) consisting of ordered pairs \((x, y)\) as defined in Section 2.2. The space \( X \times Y \) can be normed in several ways such as \( \|(x, y)\| = \|x\| + \|y\| \)
or \( \| (x, y) \| = \max \{ \| x \|, \| y \| \} \) but, unless specifically noted otherwise, we define the product norm as \( \| (x, y) \| = \| x \| + \| y \| \). It is simple to show that if \( X \) and \( Y \) are Banach spaces, the product space \( X \times Y \) with the product norm is also a Banach space.

### 2.12 Complete Subsets

The definition of completeness has an obvious extension to subsets of a normed space; a subset is complete if every Cauchy sequence from the subset converges to a limit in the subset. The following theorem states that completeness and closure are equivalent in a Banach space. This is not so in general normed space since, for example, a normed space is always closed but not necessarily complete.

**Theorem 1.** *In a Banach space a subset is complete if and only if it is closed.*

**Proof.** A complete subset is obviously closed since every convergent (and hence Cauchy) sequence has a limit in the subset. A Cauchy sequence from a closed subset has a limit somewhere in the Banach space. By closure the limit must be in the subset.

The following theorem is of great importance in many applications.

**Theorem 2.** *In a normed linear space, any finite-dimensional subspace is complete.*

**Proof.** The proof is by induction on the dimension of the subspace. A one-dimensional subspace is complete since, in such a subspace, all elements have the form \( x = \alpha e \) where \( \alpha \) is an arbitrary scalar and \( e \) is a fixed vector. Convergence of a sequence \( \alpha_n e \) is equivalent to convergence of the sequence of scalars \( \{ \alpha_n \} \) and, hence, completeness follows from the completeness of \( E \).

Assume that the theorem is true for subspaces of dimension \( N - 1 \). Let \( X \) be a normed space and \( M \) an \( N \)-dimensional subspace of \( X \). We show that \( M \) is complete.

Let \( \{ e_1, e_2, \ldots, e_N \} \) be a basis for \( M \). For each \( k \), define

\[
\delta_k = \inf_{\alpha_j} \| e_k - \sum_{j \neq k} \alpha_j e_j \|.
\]

The number \( \delta_k \) is the distance from the vector \( e_k \) to the subspace \( M_k \) generated by the remaining \( N - 1 \) basis vectors. The number \( \delta_k \) is greater than zero because otherwise a sequence of vectors in the \( N - 1 \) dimensional subspace \( M_k \) could be constructed converging to \( e_k \). Such a sequence cannot exist since \( M_k \) is complete by the induction hypothesis.
Define $\delta > 0$ as the minimum of the $\delta_k$, $k = 1, 2, \ldots, N$. Suppose that $\{x_n\}$ is a Cauchy sequence in $M$. Each $x_n$ has a unique representation as

$$x_n = \sum_{i=1}^{N} \lambda^n_i e_i.$$ 

For arbitrary $n, m$

$$\|x_n - x_m\| = \left\| \sum_{i=1}^{N} (\lambda^n_i - \lambda^m_i) e_i \right\| \geq |\lambda^n_k - \lambda^m_k| \delta$$

for each $k$, $1 \leq k \leq N$. Since $\|x_n - x_m\| \to 0$, each $|\lambda^n_k - \lambda^m_k| \to 0$. Thus, $\{\lambda^n_k\}_{n=1}^{\infty}$ is a Cauchy sequence of scalars and hence convergent to a scalar $\lambda_k$. Let $x = \sum_{k=1}^{N} \lambda_k e_k$. Obviously, $x \in M$. We show that $x_n \to x$. For all $n$, we have

$$\|x_n - x\| = \left\| \sum_{k=1}^{N} (\lambda^n_k - \lambda_k) e_k \right\| \leq N \cdot \max_{1 \leq k \leq N} |\lambda^n_k - \lambda_k| \cdot \|e_k\|,$$

but since $|\lambda^n_k - \lambda_k| \to 0$ for all $k$, $\|x_n - x\| \to 0$. Thus, $\{x_n\}$ converges to $x \in M$.

2.13 Extreme Values of Functionals and Compactness

Optimization theory is largely concerned with the maximization or minimization of real functionals over a given subset; indeed, a major portion of this book is concerned with principles for finding the points at which a given functional attains its maximum. A more fundamental question, however, is whether a functional has a maximum on a given set. In many cases the answer to this is easily established by inspection, but in others it is by no means obvious.

In finite-dimensional spaces the well-known Weierstrass theorem, which states that a continuous function defined on a closed and bounded (compact) set has a maximum and a minimum, is of great utility. Usually, this theorem alone is sufficient to establish the existence of a solution to a given optimization problem.

In this section we generalize the Weierstrass theorem to compact sets in a normed space, thereby obtaining a simple and yet general result applicable to infinite-dimensional problems. Unfortunately, however, the restriction to compact sets is so severe in infinite-dimensional normed spaces that the Weierstrass theorem can in fact only be employed in the minority of optimization problems. The theorem, however, deserves special attention in optimization theory if only because of the finite-dimensional version. The interested reader should also consult Section 5.10.
Actually, to prove the Weierstrass theorem it is not necessary to assume continuity of the functional but only upper semicontinuity. This added generality is often of great utility.

**Definition.** A (real-valued) functional \( f \) defined on a normed space \( X \) is said to be **upper semicontinuous** at \( x_0 \) if, given \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( f(x) - f(x_0) < \varepsilon \) for \( \|x - x_0\| < \delta \). A functional \( f \) is said to be **lower semicontinuous** at \( x_0 \) if \(-f\) is upper semicontinuous at \( x_0 \).

As the reader may verify, an equivalent definition is that \( f \) is upper semicontinuous at \( x_0 \) if \( 3 \limsup_{x \to x_0} f(x) \leq f(x_0) \). Clearly, if \( f \) is both upper and lower semicontinuous, it is continuous.

**Definition.** A set \( K \) in a normed space \( X \) is said to be **compact** if, given an arbitrary sequence \( \{x_i\} \) in \( K \), there is a subsequence \( \{x_{i_n}\} \) converging to an element \( x \in K \).

In finite dimensions, compactness is equivalent to being closed and bounded, but, as is shown below, this is not true in general normed space. Note, however, that a compact set \( K \) must be complete since any Cauchy sequence from \( K \) must have a limit in \( K \).

**Theorem 1.** (Weierstrass) An upper semicontinuous functional on a compact subset \( K \) of a normed linear space \( X \) achieves a maximum on \( K \).

**Proof.** Let \( M = \sup_{x \in K} f(x) \) (we allow the possibility \( M = \infty \)). There is a sequence \( \{x_i\} \) from \( K \) such that \( f(x_i) \to M \). Since \( K \) is compact, there is a convergent subsequence \( x_{i_n} \to x \in K \). Clearly, \( f(x_{i_n}) \to M \) and, since \( f \) is upper semicontinuous, \( f(x) \geq \lim f(x_{i_n}) = M \). Thus, since \( f(x) \) must be finite, we conclude that \( M < \infty \) and that \( f(x) = M \).

We offer now an example of a continuous functional on the unit sphere of \( C[0, 1] \) which does not attain a maximum (thus proving that the unit sphere is not compact).

**Example 1.** Let the functional \( f \) be defined on \( C[0, 1] \) by

\[
f(x) = \int_0^{1/2} x(t) \, dt - \int_{1/2}^1 x(t) \, dt.
\]

It is easily verified that \( f \) is continuous since, in fact, \( |f(x)| \leq \|x\| \). The supremum of \( f \) over the unit sphere in \( C[0, 1] \) is 1, but no continuous function of norm less than unity achieves this supremum. (If the problem

\( ^3 \) The \( \limsup \) of a functional on a normed space is the obvious extension of the corresponding definition for functions of a real variable.
were formulated in $L_\infty[0, 1]$, the supremum would be achieved by a function discontinuous at $t = \frac{1}{2}$.)

Example 2. Suppose that in the above example we restrict our attention to those continuous functions in $C[0, 1]$ within the unit sphere which are polynomials of degree $n$ or less. The set of permissible elements in $C[0, 1]$ is now a closed, bounded subset of the finite-dimensional space of $n$-th degree polynomials and is therefore compact. Thus Weierstrass's theorem guarantees the existence of a maximizing vector from this set.

*2.14 Quotient Spaces

Suppose we select a subspace $M$ from a vector space $X$ and generate linear varieties $V$ in $X$ by translations of $M$. The linear varieties obtained can be regarded as the elements of a new vector space called the quotient space of $X$ modulo $M$ and denoted $X/M$. If, for example $M$ is a plane through the origin in three-dimensional space, $X/M$ consists of the family of planes parallel to $M$. We formalize this definition below.

**Definition.** Let $M$ be a subspace of a vector space $X$. Two elements $x_1, x_2 \in X$ are said to be equivalent modulo $M$ if $x_1 - x_2 \in M$. In this case, we write $x_1 \equiv x_2$.

This equivalence relation partitions the space $X$ into disjoint subsets, or classes, of equivalent elements: namely, the linear varieties that are distinct translates of the subspace $M$. These classes are often called the cosets of $M$. Given an arbitrary element $x \in X$, it belongs to a unique coset of $M$ which we denote by $[x]$.

**Definition.** Let $M$ be a subspace of a vector space $X$. The quotient space $X/M$ consists of all cosets of $M$ with addition and scalar multiplication defined by $[x_1] + [x_2] = [x_1 + x_2], \alpha[x] = [\alpha x]$.

Several things, which we leave to the reader, need to be verified in order to justify the above definition. It must be shown that the definitions for addition and scalar multiplication are independent of the choice of representative elements, and the axioms for a vector space must be verified. These matters are easily proved, however, and it is not difficult to see that addition of cosets $[x_1]$ and $[x_2]$ merely amounts to addition of the corresponding linear varieties regarded as sets in $X$. Likewise, multiplication of a coset by a scalar $\alpha$ (except for $\alpha = 0$) amounts to multiplication of the corresponding linear variety by $\alpha$.

* The notation $[x]$ is also used for the subspace generated by $x$ but the usage is always clear from context.
Suppose now that $X$ is a normed space and that $M$ is a closed subspace of $X$. We define the norm of a coset $[x] \in X/M$ by

$$||[x]|| = \inf_{m \in M} ||x + m||,$$

i.e., $||[x]||$ is the infimum of the norms of all elements in the coset $[x]$. The assumption that $M$ is closed insures that $||[x]|| > 0$ if $[x] \neq \emptyset$. Satisfaction of the other two axioms for a norm is easily verified. In the case of $X$ being two dimensional, $M$ one dimensional, and $X/M$ consisting of parallel lines, the quotient norm of one of the lines is the minimum distance of the line from the origin.

**Proposition 1.** Let $X$ be a Banach space, $M$ a closed subspace of $X$, and $X/M$ the quotient space with the quotient norm defined as above. Then $X/M$ is also a Banach space.

The proof is left to the reader.

**2.15 Denseness and Separability**

We conclude this chapter by introducing one additional topological concept, that of denseness.

**Definition.** A set $D$ is said to be dense in a normed space $X$ if for each element $x \in X$ and each $\varepsilon > 0$ there exists $d \in D$ with $||x - d|| < \varepsilon$.

If $D$ is dense in $X$, there are points of $D$ arbitrarily close to each $x \in X$. Thus, given $x$, a sequence can be constructed from $D$ which converges to $x$. It follows that equivalent to the above definition is the statement: $D$ is dense in $X$ if $\overline{D}$, the closure of $D$, is $X$.

The definition converse to the above is that of a nowhere dense set.

**Definition.** A set $E$ is said to be nowhere dense in a normed space $X$ if $\overline{E}$ contains no open set.

The classic example of a dense set is the set of rationals in the real line. Another example, provided by the well-known Weierstrass approximation theorem, is that the space of polynomials is dense in the space $C[a, b]$.

**Definition.** A normed space is separable if it contains a countable dense set.

Most, but not all, of the spaces considered in this book are separable.

**Example 1.** The space $E^n$ is separable. The collection of vectors $x = (\xi_1, \xi_2, \ldots, \xi_n)$ having rational components is countable and dense in $E^n$. 


Example 2. The $l_p$ spaces, $1 \leq p < \infty$, are separable. To prove separability, let $D$ be the set consisting of all finitely nonzero sequences with rational components. $D$ is easily seen to be countable. Let $x = \{\xi_1, \xi_2, \ldots\}$ exist in $l_p$, $1 \leq p < \infty$, and fix $\varepsilon > 0$. Since $\sum_{i=1}^{\infty} |\xi_i|^p < \infty$, there is an $N$ such that $\sum_{i=N+1}^{\infty} |\xi_i|^p < \varepsilon/2$. For each $k$, $1 \leq k \leq N$, let $r_k$ be a rational such that $|\xi_k - r_k|^p < \varepsilon/2N$; let $d = \{r_1, r_2, \ldots, r_N, 0, 0, \ldots\}$. Clearly, $d \in D$ and $\|x - d\|^p < \varepsilon$. Thus, $D$ is dense in $l_p$.

The space $l_\infty$ is not separable.

Example 3. The space $C[a, b]$ is separable. Indeed, the countable set consisting of all polynomials with rational coefficients is dense in $C[a, b]$. Given $x \in C[a, b]$ and $\varepsilon > 0$, it follows from the well-known Weierstrass approximation theorem that a polynomial $p$ can be found such that $|x(t) - p(t)| < \varepsilon/2$ for all $t \in [a, b]$. Clearly, there is another polynomial $r$ with rational coefficients such that $|r(t) - p(t)| < \varepsilon/2$ for all $t \in [a, b]$ can be constructed by changing each coefficient by less than $\varepsilon/2N$ where $N - 1$ is the order of the polynomial $p$.

Thus

$$
\|x - r\| = \max_{t \in [a, b]} |x(t) - r(t)| \leq \max_{t \in [a, b]} |x(t) - p(t)| + \max_{t \in [a, b]} |p(t) - r(t)|
$$

$$
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
$$

Example 4. The $L_p$ spaces $1 \leq p < \infty$ are separable but $L_\infty$ is not separable. The particular space $L_2$ is considered in great detail in Chapter 3.

2.16 Problems

1. Prove Proposition 1, Section 2.2.
2. Show that in a vector space $(-\alpha)x = \alpha(-x) = -(\alpha x)$, $\alpha(x - y) = \alpha x - \alpha y$, $(\alpha - \beta)x = \alpha x - \beta x$.
4. A convex combination of the vectors $x_1, x_2, \ldots, x_n$ is a linear combination of the form $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ where $\alpha_i \geq 0$, for each $i$, and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. Given a set $S$ in a vector space, let $K$ be the set of vectors consisting of all convex combinations from $S$. Show that $K = \text{co}(S)$.
5. Let $C$ and $D$ be convex cones in a vector space. Show that $C \cap D$ and $C + D$ are convex cones.
6. Prove that the union of an arbitrary collection of open sets is open and that the intersection of a finite collection of open sets is open.
7. Prove that the intersection of an arbitrary collection of closed sets is closed and that the union of a finite collection of closed sets is closed.

8. Show that the closure of a set \( S \) in a normed space is the smallest closed set containing \( S \).

9. Let \( X \) be a normed linear space and let \( x_1, x_2, \ldots, x_n \) be linearly independent vectors from \( X \). For fixed \( y \in X \), show that there are coefficients \( a_1, a_2, \ldots, a_n \) minimizing \( \| y - a_1 x_1 - a_2 x_2 - \cdots - a_n x_n \| \).

10. A normed space is said to be strictly normed if \( \| x + y \| = \| x \| + \| y \| \) implies that \( y = \theta \) or \( x = \alpha y \) for some \( \alpha \).
   (a) Show that \( L_p[0, 1] \) is strictly normed for \( 1 < p < \infty \).
   (b) Show that if \( X \) is strictly normed the solution to Problem 9 is unique.

11. Prove the Hölder inequality for \( p = 1, \infty \).

12. Suppose \( x = \{ \xi_1, \xi_2, \ldots, \xi_n, \ldots \} \in l_{p_0} \) for some \( p_0, 1 \leq p_0 < \infty \).
   (a) Show that \( x \in l_p \) for all \( p \geq p_0 \).
   (b) Show that \( \| x \|_\infty = \lim_{p \to \infty} \| x \|_p \).

13. Prove that the normed space \( D[a, b] \) is complete.

14. Two vector spaces, \( X \) and \( Y \), are said to be isomorphic if there is a one-to-one mapping \( T \) of \( X \) onto \( Y \) such that \( T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2) \). Show that any real \( n \)-dimensional space is isomorphic to \( E^n \).

15. Two normed linear spaces, \( X \) and \( Y \), are said to be isometrically isomorphic if they are isomorphic and if the corresponding one-to-one mapping \( T \) satisfies \( \| T(x) \| = \| x \| \). The object of this problem is to show that any normed space \( X \) is isometrically isomorphic to a dense subset of a Banach space \( \tilde{X} \).
   (a) Let \( \tilde{X} \) be the set of all Cauchy sequences \( \{ x_n \} \) from \( X \) with addition and scalar multiplication defined coordinatewise. Show that \( \tilde{X} \) is a linear space.
   (b) If \( y = \{ x_n \} \in \tilde{X} \), define \( \| y \| = \sup \| x_n \| \). Show that with this definition, \( \tilde{X} \) becomes a normed space.
   (c) Let \( M \) be the subspace of \( \tilde{X} \) consisting of all Cauchy sequences convergent to zero, and let \( \tilde{X} = \tilde{X}/M \). Show that \( \tilde{X} \) has the required properties.

16. Let \( S, T \) be open sets in the normed spaces \( X \) and \( Y \), respectively. Show that \( S \times T = \{(s, t) : s \in S, t \in T \} \) is an open subset of \( X \times Y \) under the usual product norm.

17. Let \( M \) be an \( m \)-dimensional subspace of an \( n \)-dimensional vector space \( X \). Show that \( X/M \) is \((n - m)\) dimensional.

18. Let \( X \) be \( n \) dimensional and \( Y \) be \( m \) dimensional. What is the dimension of \( X \times Y \)?
19. A real-valued functional \(|x|\), defined on a vector space \(X\), is called a **seminorm** if:

1. \(|x| \geq 0\) for all \(x \in X\)
2. \(|\alpha x| = |\alpha| |x|\)
3. \(|x + y| \leq |x| + |y|\).

Let \(M = \{x: |x| = 0\}\) and show that the space \(X/M\) with \(\|[x]\| = \inf_{m \in M} |x + m|\) is normed.

20. Let \(X\) be the space of all functions \(x\) on \([0, 1]\) which vanish at all but a countable number of points and for which

\[ \|x\| = \sum_{n=1}^{\infty} |x(t_n)| < \infty, \]

where the \(t_n\) are the points at which \(x\) does not vanish.

(a) Show that \(X\) is a Banach space.

(b) Show that \(X\) is not separable.

21. Show that the Banach space \(l_\infty\) is not separable.

**REFERENCES**

There are a number of excellent texts on linear spaces that can be used to supplement this chapter.

§2.2. For the basic concepts of vector space, an established standard is Halmos [68].

§2.4. Convexity is an important aspect of a number of mathematical areas. For an introductory discussion, consult Eggleston [47] or, more advanced, Valentine [148].

§2.6–9. For general discussions of functional analysis, refer to any of the following. They are listed in approximate order of increasing level: Kolmogorov and Fomin [88], Simmons [139], Lusin and Sobolev [101], Goffman and Pedrick [59], Kantorovich and Akilov [79], Taylor [145], Yosida [157], Riesz and Sz-Nagy [123], Edwards [46], Dunford and Schwartz [45], and Hille and Phillips [73].

§2.10. For a readable discussion of \(l_p\) and \(L_p\) spaces and a general reference on analysis and integration theory, see Royden [131].

§2.13. For the Weierstrass approximation theorem, see Apostol [10].