Random Variable

- Let $S$ be a sample space with a probability measure $P$.
- A real random variable $x$ is a process of assigning a real number $x(s)$ to every outcome $s \in S$.
- Two types: Discrete and Continuous
Example

• Number of heads when tossing 4 coins.

\[ S = \text{the set of all possible outcomes (s)} \]
\[ x(s) = \begin{cases} 
0 & s = \{TTTT\} \\
1 & s = \{HHTT, THTT, TTHT, TTTT\} \\
2 & s = \{HHTT, HTHT, HTHH, \text{THHT, THTH, TTHH}\} \\
3 & s = \{THHH, HTHH, HHTH, HHHT\} \\
4 & s = \{HHHH\} 
\end{cases} \]

• Discrete set of values with specified probabilities
Probability-Mass Function

• If X is a discrete random variable, the function given by \( f(x) = p(x=x) \) for each \( x \) within the range of \( x \) is called the probability distribution of \( x \)

• Probability of an event is between 0 and 1

• Sum of all the probabilities is exactly 1
Expected Value

• For a discrete random variable

\[ E(x) = \mu = \sum_{i=1}^{R} x_i p(x_i) \]

• What is the expected value of a dice?
Variance

- Analogous to sample variance ($s^2$)
- For a discrete random variable ($X$) with mean $\mu$

\[
V ar(x) = \sigma^2 = E[(x - \mu)^2] = \sum_{i=1}^{R} (x_i - \mu)^2 p(x_i) \\
E[(x - \mu)^2] = \sum_{i=1}^{R} x_i^2 p(x_i) - \mu^2
\]
Binomial Distribution

• The binomial distribution of the number of successes in \( n \) independent trials where the probability of success on each trial is \( p \) and has a probability-mass function given by

\[
b(k; n, p) = \binom{n}{k} p^k (1 - p)^{n-k}
\]

\[
\sum_{k=0}^{n} b(k; n, p) = 1
\]
Poisson Distribution

• One of the fundamental distributions in modeling:
  – Examples: Number of electrons emitted by radioactive material, number of telephone calls, cars crossing a bridge…
Poisson Distribution

- Distribution of deaths over a long time period.
- Assume that probability of observing 1 death in a small time interval is directly proportional to the length of the time interval.
- The probability of observing more than one death in a small time interval is essentially 0.
- Stationary: Number of deaths per unit time interval is the same throughout the time interval.
- Independence: If a death occurs within one time interval there is no bearing on the probability of death occurring on non-intersecting time intervals.
**Poisson Distribution**

Derivation of the Poisson Points:

Place at random ("uniformly distributed") $n$ points in an interval $(-T/2, T/2)$, let $k$ be the subset of points that lie in an interval $(t_1, t_2)$ of length $t_a = (t_2 - t_1)$, Probability that there are $k$ points in the interval is given by:

$$P(k) = \binom{n}{k} \left( \frac{t_a}{T} \right)^k \left( 1 - \frac{t_a}{T} \right)^{n-k}$$
Poisson Limit Theorem

\[ \lim_{n \to \infty} \binom{n}{k} \left( \frac{a}{n} \right)^k \left(1 - \frac{a}{n} \right)^{(n-k)} = e^{-a} \left( \frac{a^k}{k!} \right) \]

\[ \binom{n}{k} (p)^k (1 - p)^{(n-k)} \approx e^{-np} \left( \frac{(np)^k}{k!} \right) \]
Poisson Points

• For large \( n \) and \( T \),

\[
P(k) \approx e^{-nt_a/T} \left( \frac{(nt_a/T)^k}{k!} \right)
\]

Now suppose that we let \( n \) and \( T \) go to infinity but keep their ratio the same

\[
\lambda = \frac{n}{T}
\]

\[
\lim_{{n \to \infty}} P(k) = e^{-\lambda t_a} \left( \frac{\lambda t_a^k}{k!} \right)
\]
Density interpretation of Poisson points

\[ \lambda \Delta t e^{-\lambda \Delta t} \approx \lambda \Delta t \]

\[ P\{\text{one point in } (t, t + \Delta t)\} \approx \lambda \Delta t \]

\[ \lambda = \lim_{\Delta t \to 0} \frac{P\{\text{one point in } (t, t + \Delta t)\}}{\Delta t} \]