Random Variable

- Let $S$ be a sample space with a probability measure $P$.
- A real random variable $x$ is a process of assigning a real number $x(s)$ to every outcome $s \in S$.
- Two types: Discrete and Continuous
Distribution Function

• Definition: The **Distribution Function** of the random variable $\mathbf{x}$ is the function

$$F_x(\mathbf{x}) = P(\mathbf{x} \leq \mathbf{x})$$
Example (Discrete)

• Dice:

\[
\begin{align*}
F(x) &= 0 \quad \text{for } x < 1 \\
F(x) &= \frac{1}{6} \quad \text{for } 1 \cdot x < 2 \\
F(x) &= \frac{2}{6} \quad \text{for } 2 \cdot x < 3 \\
F(x) &= \frac{3}{6} \quad \text{for } 3 \cdot x < 4 \\
F(x) &= \frac{4}{6} \quad \text{for } 4 \cdot x < 5 \\
F(x) &= \frac{5}{6} \quad \text{for } 5 \cdot x < 6 \\
F(x) &= \frac{6}{6} \quad \text{for } 6 \cdot x
\end{align*}
\]
Properties of Distribution Function

\[ F(\infty) = 1 \quad F(-\infty) = 0 \]

If \( x_1 < x_2 \), then \( F(x_1) \leq F(x_2) \)

If \( F(x_0) = 0 \), then \( F(x) = 0 \) for every \( x \leq x_0 \)

\[ P(\{x > x\}) = 1 - F(x) \]

\[ P(\{x_1 < x \leq x_2\}) = F(x_2) - F(x_1) \]

\[ P(\{x = x\}) = F(x) - F(x^-) \]

\[ P(\{x_1 \leq x \leq x_2\}) = F(x_2) - F(x_1^-) \]
Example (Continuous RV)

- A telephone call occurs at a random interval in (0,1). In this experiment, the outcomes are the time instances $t$ between 0 and 1 and the probability that $t$ is between $t_1$ and $t_2$ is given by

$$P(t_1 \leq t \leq t_2) = t_2 - t_1$$

Define a random variable $x$ such that

$$x(t) = t$$
Continuous, discrete and mixed type

• We shall say that a random variable $x$ is of continuous type if its distribution function $F(x)$ is continuous, $F(x^-)= F(x)$, hence $P(\{x=x\}) = 0$

• We shall say that $x$ is discrete type if $F(x)$ is a staircase function. Denoting $x_i$ the discontinuity points of $F(x)$, we have $P(\{x=x_i\}) = F(x_i-x_i^-)=p_i$

• We shall say that $x$ is of mixed type if $F(x)$ is discontinuous but not a staircase.

• If the sample space $S$ has finitely many elements, then any random variable is of discrete type. However a random variable might be of discrete type even if $S$ has infinitely many elements.
The Density function

• The Derivative

\[ f_x(x) = \frac{dF_x(x)}{dx} \]

Is called the density function.

• If the random variable is of discrete type taking values \( x_i \) with probabilities \( p_i \) than the density function is given by

\[ f(x) = \sum_{i} p_i \delta(x - x_i) \]
Properties of density function

\[ f(x) \geq 0 \]

\[ F(x) = \int_{-\infty}^{x} f(t) \, dt \]

\[ \int_{-\infty}^{\infty} f(x) \, dx = 1 \]

\[ F(x_2) - F(x_1) = \int_{x_1}^{x_2} f(x) \, dx \]
Mathematical Definition

• For a discrete random variable $x$ the expected value $E(x)$ of the random variable is defined as:

$$E(x) = \sum_{x} x P(x)$$

• For a continuous random variable the expected value is defined as:

$$E(x) = \int_{-\infty}^{\infty} x f(x) \, dx$$
Expectation of a function of a Random Variable

• For any function $g(x)$ the Expected value is defined as

$$E\{g(x)\} = \int_{-\infty}^{\infty} g(x) f(x) \, dx$$